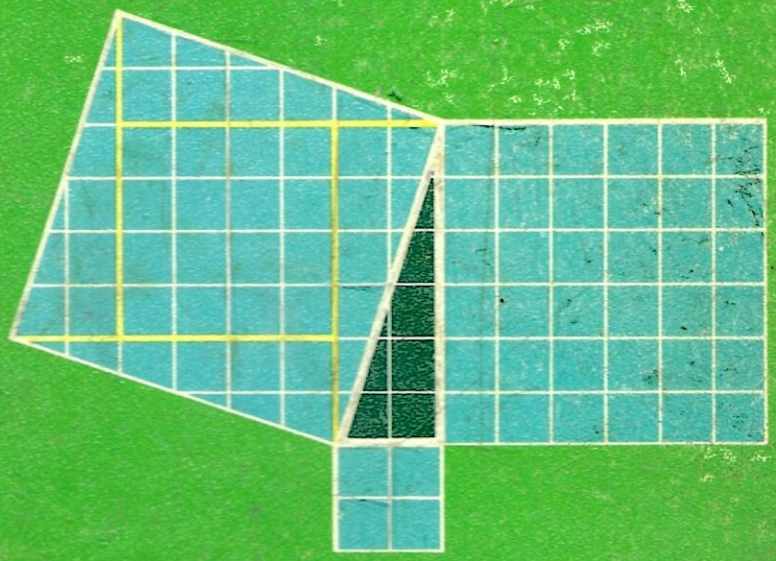
SCHOOL MATHEMATICS

I



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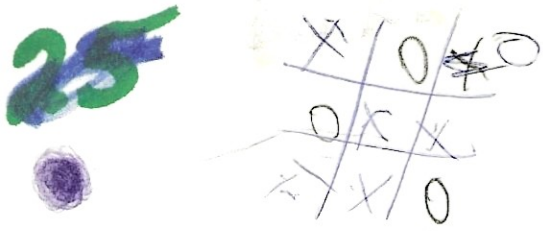

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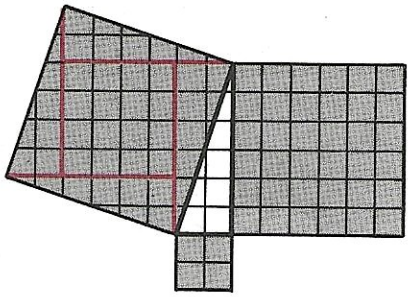

I

SCHOOL MATHEMATICS

ROBERT E. EICHOLZ
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CHARLES R. FLEENOR

CALIFORNIA STATE SERIES
Published by
CALIFORNIA DEPARTMENT OF EDUCATION
Sacramento, 1970

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Chapter Openings

Chapter frontispiece quotations are taken from Lewis Carroll's *Alice in Wonderland* and *Through the Looking Glass.* Frontispiece illustrations are adapted from John Tenniel's drawings for Lewis Carroll's books.

Lewis Carroll, whose actual name was Charles Dodgson, was an accomplished mathematician as well as a writer, and his fictional works (such as the two named above) frequently use mathematical ideas in a humorous and paradoxical manner. Consequently, the chapter frontispiece quotations are chosen not only for their humor, but also for their relevance to the material in the chapter.

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To the Teacher

In School Mathematics I *and* II, *as in the* Elementary School Mathematics *series, there is a twofold purpose. The entire program is committed to the encouragement of an enthusiastic curiosity about the basic concepts and structure of mathematics and to the development of proficiency in mathematical skills, including an understanding of their application to our contemporary, quantitative world.*

Each text emphasizes the discovery approach to learning, often utilizing a nonverbal presentation of mathematics through diagrams, charts, and sequences of discovery activities. Flexibility is an important feature of each text. The unusual variety of standard materials, as well as new topics, enrichment materials, and exercises graduated in difficulty, provides an appeal for students of widely divergent abilities and interests.

Above all, the authors intend that School Mathematics I *and* II *provide the atmosphere and the tools for an exciting adventure in mathematics, developing the student's ability to move ahead and attack new problems with confidence.*

To the Student

You will notice from the beginning that we are very much concerned in this text with the "why" of things. Much stress is placed upon understanding concepts and discovering new mathematical ideas. While it is important that you develop skill in mathematics, you will also need to know why various procedures work, as well as when and where you should use them. Perhaps the most important thing you can gain from this text is the ability to attack new problems and think for yourself.

Study carefully. As you study, keep pencil and paper at hand. Work out the examples to make sure you understand the ideas. Ask questions in class when you do not thoroughly understand what is being discussed. There are many challenging ideas in this book that you will enjoy mastering. Your real reward will come when you know that you have learned and understood an important new idea.

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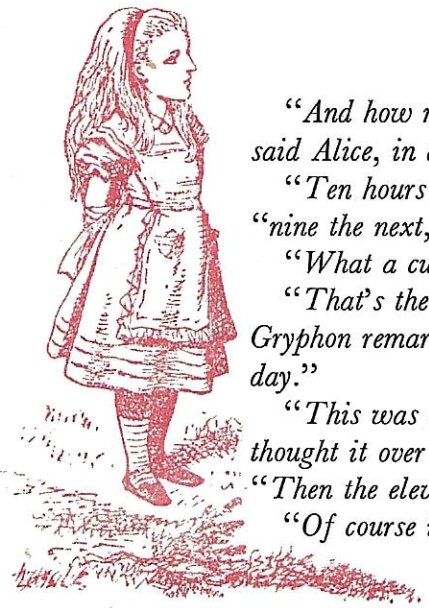


"And how many hours a day did you do lessons?" said Alice, in a hurry to change the subject.

"Ten hours the first day," said the Mock Turtle: "nine the next, and so on."

"What a curious plan!" exclaimed Alice.

"That's the reason they're called lessons," the Gryphon remarked: "because they lessen from day to day."

"This was quite a new idea to Alice, and she thought it over a little before she made her next remark. "Then the eleventh day must have been a holiday?"

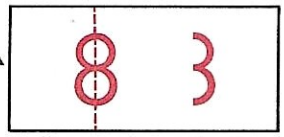
"Of course it was," said the Mock Turtle.

1

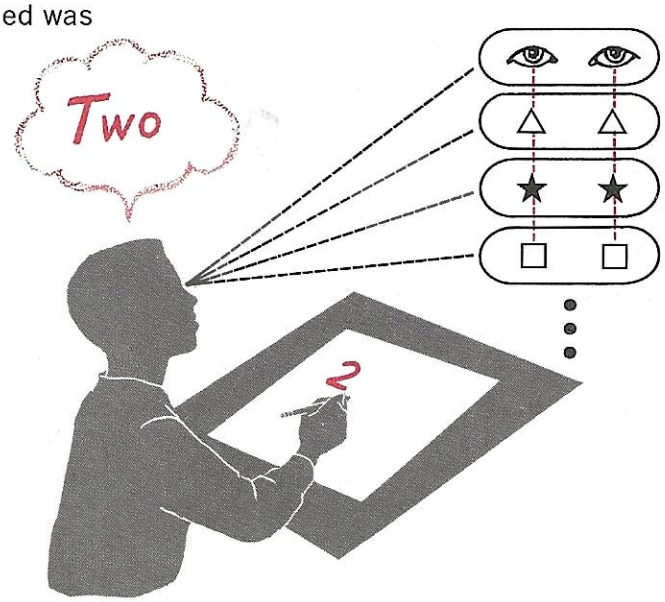
Numbers and Numeration Systems

1-1 SETS, NUMBERS AND NUMERALS

Ned said, "Half of 8 is 3," and wrote this to prove it. Fred said, "No, half of 8 is 4." Who do you think was correct? Why?



Fred was talking about numbers. Ned was talking about numerals. A **number** is an idea. For example, when we think about sets (groups) of objects that can be matched one to one with a pair of eyes, we are thinking about the number two.



A **numeral** is usually something we write. It is a symbol (mark) or collection of symbols representing a number.

A **numeration system** is a system for writing numerals. Our numeration system uses ten special symbols, called **digits**, to represent whole numbers. The digits are shown in this table.

We see sets of objects that can be matched one to one.	We **think** about a **whole number.**	We **write** a **numeral** (digit) for the number.
{★} {▲} {●} . . .	one	1
{★★} {▲▲} {●●} . . .	two	2
{★★★} {▲▲▲} {●●●} . . .	three	3
{★★ / ★★} {▲▲ / ▲▲} {●● / ●●} . . .	four	4
{★ ★ / ★ ★} {▲ ▲ / ▲ ▲} {● ● / ● ●} . . .	five	5
{★★★ / ★★★} {▲▲▲ / ▲▲▲} {●●● / ●●●} . . .	six	6
{★★★ / ★★★★} {▲▲▲ / ▲▲▲▲} {●●● / ●●●●} . . .	seven	7
{★★★★ / ★★★★} {▲▲▲▲ / ▲▲▲▲} {●●●● / ●●●●} . . .	eight	8
{★★★ / ★★★ / ★★★} {▲▲▲ / ▲▲▲ / ▲▲▲} {●●● / ●●● / ●●●} . . .	nine	9
{ }	zero	0

To indicate a set which has no objects in it, as in the last line of the table, we write { }. We call this set the **empty set**.

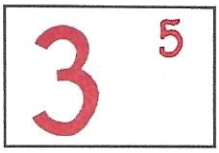
Exercises

1. A set of objects is a collection or group of objects. Write digits that show how many objects are in each set.

[**a**] The set of wheels on a bicycle
[**b**] The set of fingers on your hand
[**c**] The set of letters used in the word "numeral"
[**d**] The set of blue buffaloes in your classroom
[**e**] The set of whole numbers less than 10 and greater than 6
[**f**] The set of singers in a trio
[**g**] The set of whole numbers larger than 10 and less than 20
[**h**] The set of students sitting on the ceiling

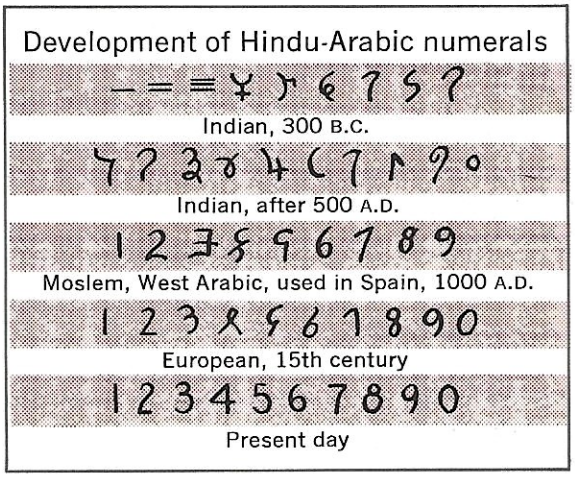
2. Write the correct word (**number**, **numeral**) for each blank.

[**a**] A _?_ is any symbol that stands for a number.
[**b**] A _?_ is an idea that is represented by a numeral.
[**c**] You write a _?_.
[**d**] You think about a _?_.



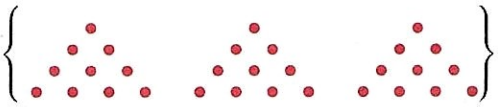
3. [**a**] If you look at the figure to the right and think about **numbers**, which is larger?
[**b**] If you think about **numerals**, which is larger?

4. The numerals we usually use are called Hindu-Arabic numerals because they were first used by the Hindus and introduced into Europe by the Arabs. The development of Hindu-Arabic numerals is shown in the display to the right. In the exercises that follow, use combinations of two digits to write a Hindu-Arabic numeral that tells how many dots are in each set.

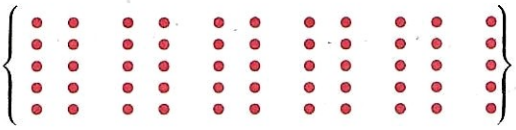

Development of Hindu-Arabic numerals
Indian, 300 B.C.
Indian, after 500 A.D.
Moslem, West Arabic, used in Spain, 1000 A.D.
European, 15th century
1 2 3 4 5 6 7 8 9 0
Present day

[**a**]

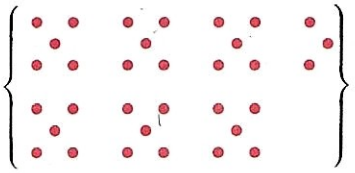
[**b**]



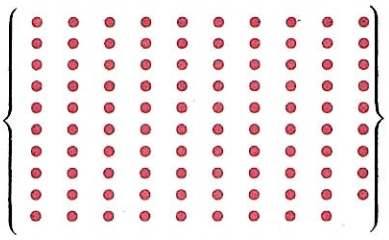
[**c**]



[**d**]



[**e**]



5. If one more dot were in the set in Exercise 4**e**, what numeral would represent the number of dots?

6. It is often helpful to think of the printing on Card A, for example, as a numeral representing just one number. Since the printing on Card B **represents** the same number, we write $9 + 6 = 20 - 5$. Since the printing on Card C does not represent the same number as the printing on Card A, we write $9 + 6 \neq 15 \times 0$.

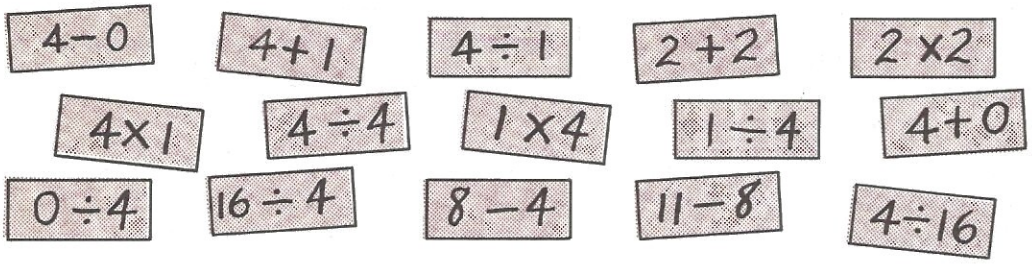
[a] Write five different numerals for the number 20.

[b] Consider the marks on each card as a numeral for just one number. Which of the following are not numerals for the number 4?

Card A

Card B

Card C



★ **7.** Using the symbol "+" once, each of the digits 1, 3, and 4 once, and using no other symbol, write a numeral for each of the following:

[a] 35 [b] 17 [c] 44

Think

1. Copy each exercise and insert any of the symbols $+, \times, -, \div,$ or $()$ in order to make the sentence true.

Example: $2 = (4 + 4) \div 4$

[a] $0 = 4\ \ 4\ \ 4$ [c] $4 = 4\ \ 4\ \ 4$ [e] $12 = 4\ \ 4\ \ 4$ [g] $32 = 4\ \ 4\ \ 4$

[b] $3 = 4\ \ 4\ \ 4$ [d] $5 = 4\ \ 4\ \ 4$ [f] $20 = 4\ \ 4\ \ 4$ [h] $64 = 4\ \ 4\ \ 4$

2. Copy each exercise and insert any of the symbols $+, \times, -, \div, (), !,$ or $\sqrt{\ }$ in order to make the sentence true.

$\sqrt{4} = 2$ (Read $\sqrt{4}$ as "the square root of 4.")

$4! = 4 \times 3 \times 2 \times 1$ (Read 4! as "4 factorial.")

[a] $1 = 4\ \ 4\ \ 4$ [b] $6 = 4\ \ 4\ \ 4$ [c] $8 = 4\ \ 4\ \ 4$ [d] $25 = 4\ \ 4\ \ 4$

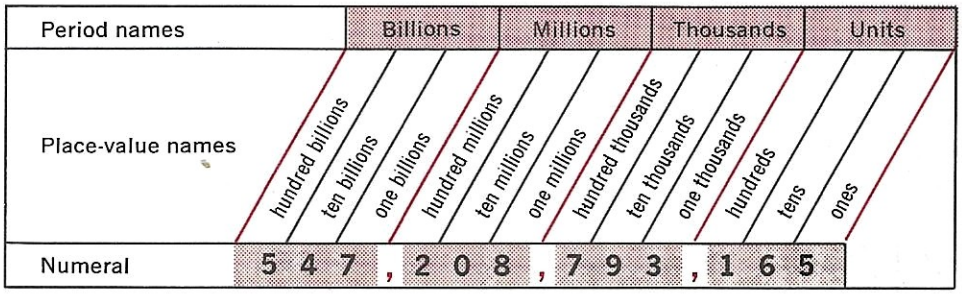
1-2 PLACE VALUE—BASE TEN

Using **place value**, we can write the numeral for any whole number, using only the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

In our numeration system, we **group by tens**. Depending upon its place in a numeral, a digit may tell how many ones, how many tens, how many groups of 10 tens, and so on.

6543 ⟶ The 3 tells how many ones. ⟶ (3×1)
6534 ⟶ The 3 tells how many tens. ⟶ (3×10)
6354 ⟶ The 3 tells how many hundreds. ⟶ (3×100)
3654 ⟶ The 3 tells how many thousands. ⟶ (3×1000)

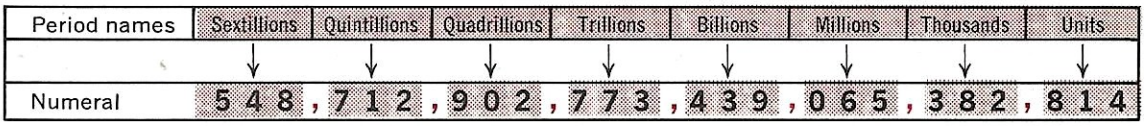
To write numerals with several digits, we separate the digits into groups of three: each group is called a **period**. The diagram shows periods and place value through the billions.



Oral Exercises

1. Read the numeral in the diagram above.

2. The diagram below shows the names of some of the periods for numbers larger than a billion. Read the numeral given in this diagram.



3. Read each fact.

[a] Earth's weight: 6,600,000,000,000,000,000,000 tons

[b] The distance light travels in a year: 5,878,496,538,000 miles (mi)

[c] Estimated age of the sun: 100,000,000,000,000,000 seconds (sec)

Exercises

1. Each sentence is about a digit in the numeral **574,896,321.**
Copy the sentences. Give the missing words and numbers.

[a] 2 in the tens place means 2×10.
[b] 9 in the _?_ place means $9 \times$ ▒.
[c] 3 in the _?_ place means $3 \times$ ▒.
[d] 5 in the _?_ place means $5 \times$ ▒.
[e] 6 in the _?_ place means $6 \times$ ▒.
[f] 4 in the _?_ place means $4 \times$ ▒.

2. In the example, the number is written using **expanded notation.**
Use expanded notation to represent each number.

Example: $7423 = (7 \times 1000) + (4 \times 100) + (2 \times 10) + 3$

[a] 358
[b] 4562
[c] 53,729
[d] 70,364
[e] 458,297
[f] 6,200,876
[g] 24,364,200
[h] 647,870,000

3. Each number is written using expanded notation. Give the regular numeral for each number.

[a] $(9 \times 100) + (7 \times 10) + 4$
[b] $(8 \times 1000) + (6 \times 100) + (7 \times 10) + 5$
[c] $(5 \times 10{,}000) + (3 \times 1000) + (4 \times 100) + (2 \times 10) + 7$
[d] $(3 \times 100{,}000) + (5 \times 1000) + (8 \times 100) + (9 \times 10) + 4$

4. Solve the equations. Part **a** is completed as an example.

[a] $10 \times 10 = n$ $(n = 100)$
[b] $10 \times 100 = n$
[c] $10 \times (10 \times 10) = n$
[d] $10 \times 1000 = n$
[e] $10 \times (10 \times 10 \times 10) = n$
[f] $10 \times 10 \times 10 \times 10 \times 10 = n$

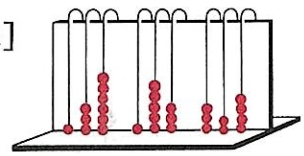
5. Write the usual Hindu-Arabic numeral for each number.

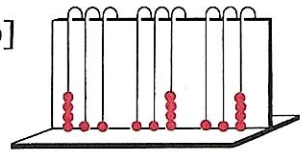
[a] $(3 \times 10 \times 10) + (7 \times 10) + 2$
[b] $(5 \times 10 \times 10) + 9$
[c] $(1 \times 10 \times 10) + (1 \times 10) + 1$
[d] $(4 \times 10 \times 10 \times 10) + (5 \times 10 \times 10) + (6 \times 10) + 7$
[e] $(9 \times 10 \times 10 \times 10) + (9 \times 10) + 9$
[f] $(6 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + 1$
[g] $(8 \times 10 \times 10 \times 10) + 8$
[h] $(3 \times 10 \times 10 \times 10 \times 10) + (4 \times 10 \times 10 \times 10) + (5 \times 10 \times 10) + (6 \times 10) + 7$
[i] $(6 \times 10 \times 10 \times 10 \times 10 \times 10) + (9 \times 10 \times 10) + (4 \times 10)$
[j] $(4 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10) + (7 \times 10 \times 10 \times 10) + 3$

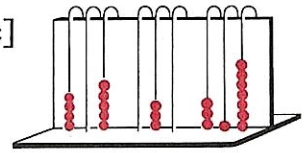
6. Use expanded notation, as shown in Exercise 5, to represent each number.

[a] 524	[d] 4070	[g] 90,909	[j] 500,500
[b] 302	[e] 5500	[h] 40,600	[k] 4,300,001
[c] 5003	[f] 73,000	[i] 380,004	[l] 5,050,505

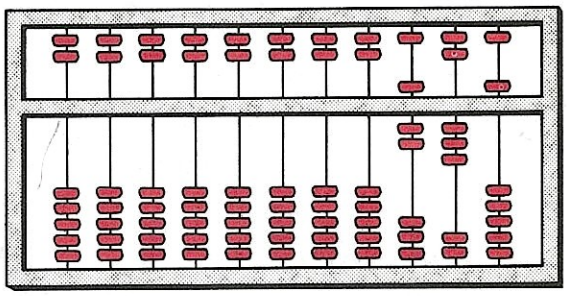
★**7.** An **abacus** is a simple computing device that uses place value. Give the number shown by each abacus.

[a] 

[b] 

[c] 

★**8.** What number is shown on the Chinese abacus?



A Chinese abacus

★**9.** How many different Hindu-Arabic numerals can you write that have

[a] just one digit
[b] not more than two digits
[c] not more than three digits
[d] not more than four digits

Think

1. Suppose you had **one billion** dollars to give away. If you give away a $100 bill each minute, how long will it take you to give it all away? Guess which answer is closest to the correct answer.

[a] A day [b] A week [c] A month [d] A year [e] 10 years [f] 20 years [g] A century [h] 2000 years

2. Compute the correct answer to the nearest year.

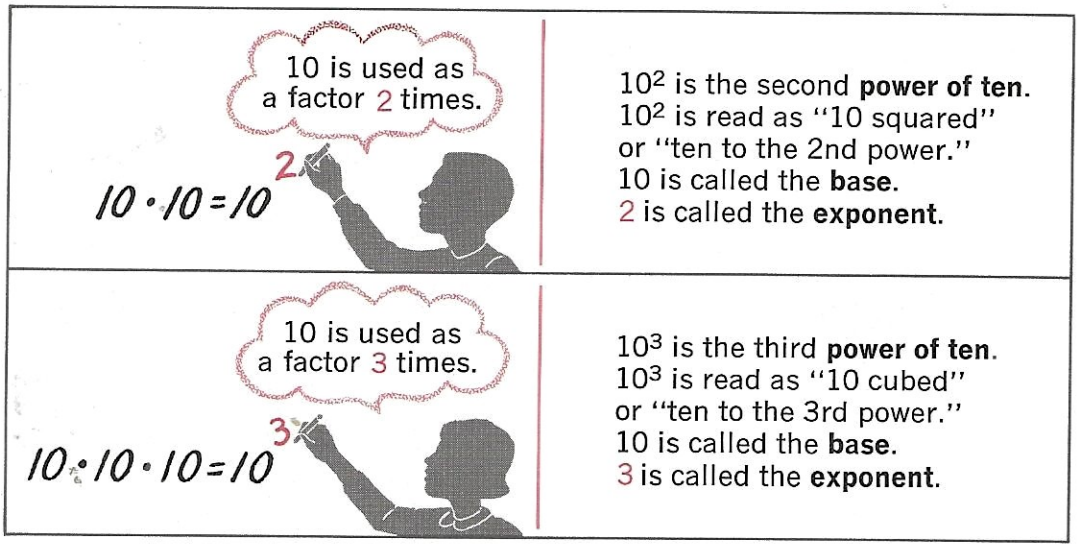
1-3 EXPONENTS

Numbers that are multiplied together to give a product are called **factors** of the product. Instead of writing "×" to show multiplication of factors, we can often use a centered dot " "—instead of the above equation, we write: When the factors are the same, we can use **exponents** to simplify the writing.

$10 \times 10 = 100$

Factors: 10, 10 — Product: 100

$10 \cdot 10 = 100$



10^2 is the second **power of ten.**
10^2 is read as "10 squared" or "ten to the 2nd power."
10 is called the **base.**
2 is called the **exponent.**

10^3 is the third **power of ten.**
10^3 is read as "10 cubed" or "ten to the 3rd power."
10 is called the **base.**
3 is called the **exponent.**

The first power of ten, 10^1, is usually written as "10."

Exercises

1. Which of the statements are false?

[a] $10^2 = 10 \cdot 10$ [b] $10^2 = 2 \cdot 10$ [c] $10^2 = 100$ [d] $10^3 = 10 \cdot 10 \cdot 10$ [e] $10^3 = 10 \cdot 3$ [f] $10^3 = 1000$ [g] $10 \cdot 100 = 10^4$ [h] $10^4 = 10 \cdot 10^3$

2. Give the missing words and numbers.

[a] 10^4 means 10 used as a factor _?_ times $\rightarrow 10^4 = 10 \cdot 10 \cdot 10 \cdot 10 =$ ▒.
[b] 10^5 means 10 used as a factor _?_ times $\rightarrow 10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ ▒.
[c] 10^6 means 10 used as a factor _?_ times $\rightarrow 10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ ▒.

3. Write each of the following, using 10 with an exponent.

[a] $10 \cdot 10 \cdot 10$ [b] $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ [c] $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

4. How many tens are used as factors in each number?

[a] 10^2 [b] 10^3 [c] 10^8 [d] 10^{19} [e] 10^6 [f] 10^{72}
[g] 10^{100} (This number is sometimes called a **googol**.)

5. Give the number for each ▓.

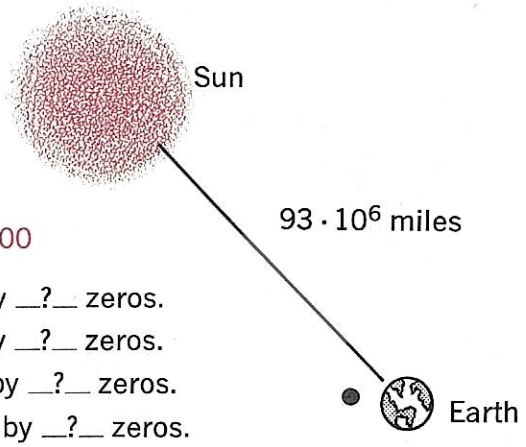
[a] $10^2 =$ ▓ [c] $10^4 =$ ▓ [e] $10^6 =$ ▓ [g] $10^8 =$ ▓
[b] $10^3 =$ ▓ [d] $10^5 =$ ▓ [f] $10^7 =$ ▓ [h] $10^9 =$ ▓

6. Find the products.

[a] $4 \cdot 10^2$ [b] $8 \cdot 10^4$ [c] $7 \cdot 10^3$ [d] $6 \cdot 10^6$ [e] $9 \cdot 10^5$ [f] $5 \cdot 10^7$

7. Check the products. Then give the correct number of zeros for each blank.

$10^1 = 10$
$10^2 = 10 \cdot 10 = 100$
$10^3 = 10 \cdot 10 \cdot 10 = 1000$
$10^4 = 10 \cdot 10 \cdot 10 \cdot 10 = 10,000$
$10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000$
$10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 1,000,000$



[a] To write 10^3, we can write a 1 followed by __?__ zeros.
[b] To write 10^6, we can write a 1 followed by __?__ zeros.
[c] To write 10^{19}, we can write a 1 followed by __?__ zeros.
[d] To write 10^{100}, we can write a 1 followed by __?__ zeros.

8. Write each of the following, using 10 with an exponent.

[a] 100 [b] 1000 [c] 10,000 [d] 100,000 [e] 1,000,000

[f] 100,000,000,000,000,000 (This number is the estimated age of the sun in seconds.)

[g] 100,000,000,000,000,000,000,000,000 (According to Einstein, this number represents the possible radius, in feet, of the universe.)

[h] 10,000,000,000,000,000,000,000,000 (This number is the approximate weight of the earth in pounds.)

9. Find the products.

[a] $10^1 \cdot 10^2$
[b] $10^2 \cdot 10^2$
[c] $10^3 \cdot 10^1$
[d] $10^3 \cdot 10^2$
[e] $10^2 \cdot 10^3$
[f] $10^3 \cdot 10^3$
[g] $53 \cdot 10^1$
[h] $276 \cdot 10^1$
[i] $24 \cdot 10^2$
[j] $365 \cdot 10^2$
[k] $42 \cdot 10^3$
[l] $57 \cdot 10^3$
[m] $87 \cdot 10^1$
[n] $935 \cdot 10^2$
[o] $68 \cdot 10^2$
[p] $793 \cdot 10^3$

1-4 MORE ABOUT EXPONENTS

Discussion Exercises

1. In the last lesson, perhaps you discovered a shortcut for writing the usual numeral for a power of ten. What does the exponent tell you about the number of zeros to write after the 1?

2. In the table at the right, by what must you multiply each number to get the number directly below it?

3. According to the pattern in the table, what do you think we should mean by 10^0?

A	B
$10^0 =$	?
$10^1 =$	10
$10^2 =$	100
$10^3 =$	1000
$10^4 =$	10,000
$10^5 =$	100,000
$10^6 =$	1,000,000
⋮	⋮

It is often helpful to use an exponent form of expanded notation. For example, to write the numeral **563** in expanded notation (exponent form), we write $(\mathbf{5} \cdot 10^2) + (\mathbf{6} \cdot 10^1) + (\mathbf{3} \cdot 10^0)$.

Exercises

1. Each expression below is written in expanded notation (exponent form). Write the usual Hindu-Arabic numeral for each expression.

[a] $(4 \cdot 10^2) + (5 \cdot 10^1) + (6 \cdot 10^0)$
[b] $(6 \cdot 10^2) + (4 \cdot 10^1) + (3 \cdot 10^0)$
[c] $(5 \cdot 10^3) + (7 \cdot 10^2) + (3 \cdot 10^1) + (2 \cdot 10^0)$
[d] $(8 \cdot 10^3) + (0 \cdot 10^2) + (4 \cdot 10^1) + (9 \cdot 10^0)$
[e] $(3 \cdot 10^5) + (7 \cdot 10^4) + (2 \cdot 10^3) + (5 \cdot 10^2) + (4 \cdot 10^1) + (9 \cdot 10^0)$
[f] $(8 \cdot 10^6) + (9 \cdot 10^5) + (7 \cdot 10^4) + (3 \cdot 10^3) + (5 \cdot 10^2) + (1 \cdot 10^1) + (6 \cdot 10^0)$
[g] $(9 \cdot 10^9) + (6 \cdot 10^6) + (3 \cdot 10^3) + (1 \cdot 10^0)$

2. Write a power of ten for n.

[a] $200 = 2 \cdot 100 = 2 \cdot n$
[b] $30 = 3 \cdot 10 = 3 \cdot n$
[c] $5000 = 5 \cdot 1000 = 5 \cdot n$
[d] $40{,}000 = 4 \cdot 10{,}000 = 4 \cdot n$
[e] $800{,}000 = 8 \cdot 100{,}000 = 8 \cdot n$
[f] $9{,}000{,}000 = 9 \cdot 1{,}000{,}000 = 9 \cdot n$

3. Write each of the following as a number less than ten multiplied by a power of ten. Example: For 300, write $3 \cdot 10^2$

[a] 600 [b] 3000 [c] 70,000 [d] 400,000 [e] 8,000,000

4. Use expanded notation (exponent form) to represent each number.

[a] 435 [b] 236 [c] 602 [d] 5627 [e] 5034 [f] 5609 [g] 8970 [h] 53,427 [i] 67,038 [j] 8,423,762

5. For each exercise, give the product **P**. Then give the missing power of ten for each ▒.

[a] $10^2 \cdot 10^1 = ▒$
$100 \cdot 10 = \mathbf{P}$

[b] $10^2 \cdot 10^2 = ▒$
$100 \cdot 100 = \mathbf{P}$

[c] $10^1 \cdot 10^3 = ▒$
$10 \cdot 1000 = \mathbf{P}$

[d] $10^3 \cdot 10^2 = ▒$
$1000 \cdot 100 = \mathbf{P}$

[e] $10^4 \cdot 10^1 = ▒$
$10{,}000 \cdot 10 = \mathbf{P}$

[f] $10^3 \cdot 10^0 = ▒$
$1000 \cdot 1 = \mathbf{P}$

6. For each exercise, give the quotient **Q**. Then give the missing power of ten for each ▒.

[a] $10^2 \div 10^1 = ▒$
$100 \div 10 = \mathbf{Q}$

[b] $10^3 \div 10^1 = ▒$
$1000 \div 10 = \mathbf{Q}$

[c] $10^3 \div 10^2 = ▒$
$1000 \div 100 = \mathbf{Q}$

[d] $10^6 \div 10^3 = ▒$
$1{,}000{,}000 \div 1000 = \mathbf{Q}$

[e] $10^5 \div 10^2 = ▒$
$100{,}000 \div 100 = \mathbf{Q}$

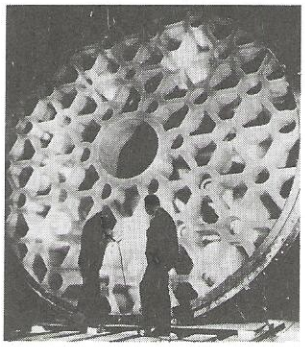
[f] $10^3 \div 10^3 = ▒$
$1000 \div 1000 = \mathbf{Q}$

★**7.** Write each product or quotient as a power of ten.
In Exercises 5 and 6, you may have discovered a shortcut that will make your work easy.

[a] $10^5 \cdot 10^4$ [b] $10^{20} \cdot 10^{15}$ [c] $10^{25} \cdot 10^5$ [d] $10^{10} \cdot 10^8$ [e] $10^6 \div 10^2$

8. Exponents are used to simplify the writing of large numbers. Each number in column A is represented in column B using **scientific notation**. A number is represented using scientific notation when it is written as a **number between 1 and 10** times a power of ten. For each sentence below, write the number using scientific notation.

A		B
200 =	$2 \cdot 100$	$= 2 \cdot 10^2$
5000 =	$5 \cdot 1000$	$= 5 \cdot 10^3$
7,000,000 =	$7 \cdot 1{,}000{,}000$	$= 7 \cdot 10^6$



Used in the Hale telescope, this 200-in. lens is the largest single piece of glass ever made.

[a] A jet airliner can travel 600 miles per hour (mph).

[b] A giant telescope lens has a 200-inch (in.) diameter.

[c] A large elephant weighs about 9000 pounds (lb).

[d] An artificial satellite travels less than 20,000 mph.

[e] A whale weighs 200,000 lb.

[f] The diameter of the earth is about 40,000,000 feet (ft).

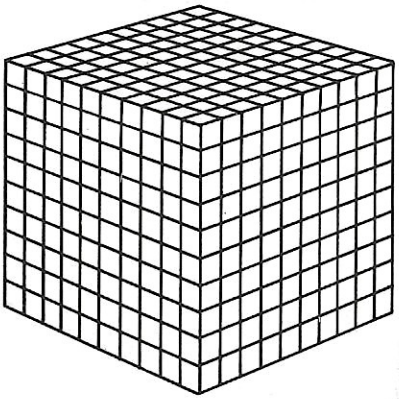
[g] The diameter of the sun is about 5,000,000,000 ft.

[h] Light travels about 30,000,000,000,000,000 ft per year (yr).

[i] The sun weighs about 4,000,000,000,000,000,000,000,000,000,000 lb.

Think

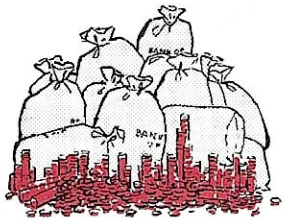
Suppose you have a solid 10 by 10 by 10-in. cube with 1-in. squares marked on each side.



1. If you have a knife and make straight cuts completely through the cube, what is the fewest number of cuts needed to divide the cube into 1-in. cubes? (Assume that the cube retains its shape after a cut.)
2. How many cubes will there be?

1-5 APPROXIMATION AND ROUNDING

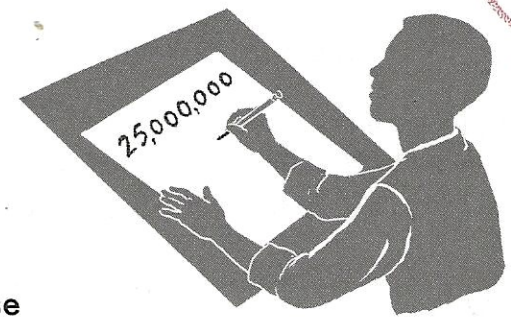
During a recent year, $25,281,314 worth of pennies were manufactured in the United States. When we do not need completely accurate information, we often use a number that is "close" or approximately equal to a given number. We say that the number of dollars worth of pennies produced was approximately



25,281,310 (rounded to the nearest ten)
or 25,281,300 (rounded to the nearest hundred)
or 25,281,000 (rounded to the nearest thousand)
or 25,280,000 (rounded to the nearest ten thousand)
or 25,300,000 (rounded to the nearest hundred thousand)
or 25,000,000 (rounded to the nearest million)
or 30,000,000 (rounded to the nearest ten million)

Study the example to see how you might think about rounding 25,281,314 to the nearest million.

25,281,314



Since 2 is less than halfway from zero to 10, I won't increase the number of millions.

Discussion Exercise

Explain how we might think about each of the other approximations given above.

Exercises

1. Choose the number closest to the number in color. If the number is halfway, choose the larger number.

[a] 60; 64 ; 70 [b] 80; 86 ; 90 [c] 300; 350 ; 400
[d] 700; 778 ; 800 [e] 3000; 3623 ; 4000 [f] 8000; 8499 ; 9000
[g] 50,000; 52,965 ; 60,000 [h] 10,000; 14,999 ; 20,000
[i] 400,000; 451,600 ; 500,000 [j] 200,000; 248,771 ; 300,000
[k] 6,000,000; 6,399,999 ; 7,000,000 [l] 9,000,000; 9,764,862 ; 10,000,000

2. Choose the number that best completes each sentence.

[a] 49,384 rounded to the nearest ten is $\begin{cases} 49{,}380. \\ 49{,}390. \end{cases}$

[b] 49,384 rounded to the nearest hundred is $\begin{cases} 49{,}300. \\ 49{,}400. \end{cases}$

[c] 49,384 rounded to the nearest thousand is $\begin{cases} 49{,}000. \\ 50{,}000. \end{cases}$

[d] 49,384 rounded to the nearest ten thousand is $\begin{cases} 40{,}000. \\ 50{,}000. \end{cases}$

[e] 7,457,000 rounded to the nearest hundred thousand is $\begin{cases} 7{,}400{,}000. \\ 7{,}500{,}000. \end{cases}$

[f] 56,501,276 rounded to the nearest million is $\begin{cases} 56{,}000{,}000. \\ 57{,}000{,}000. \end{cases}$

3. The earth's average diameter (distance through) is about 7918 mi. Give this number of miles

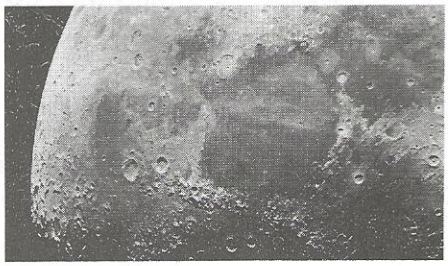
[a] to the nearest hundred [b] to the nearest thousand

4. The earth's circumference (distance around) at the equator is about 24,902 mi. Give this number of miles

[a] to the nearest thousand [b] to the nearest ten thousand

5. The average distance from the earth to the moon is 238,857 mi. Give this number of miles

[a] to the nearest hundred
[b] to the nearest thousand
[c] to the nearest hundred thousand



A portion of the moon's surface

6. The average distance from the earth to the sun is 92,955,750 mi. Give this number of miles

[a] to the nearest hundred
[b] to the nearest hundred thousand
[c] to the nearest million

7. The surface area of the earth is 196,940,400 square miles (sq mi). Give this area to the nearest hundred million.

★**8.** The diameter of the sun is about 109 times the diameter of the earth. (See Exercise 3.)

[**a**] Find the diameter of the sun and round it to the nearest hundred thousand miles.

[**b**] Write this number, using scientific notation.

★**9.** A ball's volume (space taken up) can be measured approximately by finding the third power of the diameter (d^3 or $d \cdot d \cdot d$) and dividing this number by 2.

[**a**] Using your answer to Exercise 3**b**, find the approximate volume of the earth.

[**b**] Using your answer to Exercise 8**a**, find the approximate volume of the sun.

[**c**] About how many "earths" would it take to fill as much space as the sun fills?

1-6 INEQUALITIES

We often use the **inequality symbols** $>$ and $<$ to state the comparison between the number of objects in two sets. Your knowledge of place value will help you decide which of two numbers is larger.

{★★★★★★★★★★ ★★★★★★}
⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮
{××××××××××× ××××}

To abbreviate 16 is greater than 14, we write: $16 > 14$.

To abbreviate 14 is less than 16, we write: $14 < 16$.

Discussion Exercise

What are some good ways to remember which symbol, $>$ or $<$, means "greater than" and which means "less than"?

Exercises

1. Give the correct symbol ($<$, $>$, or $=$) for each ◍.

[**a**] $3654 \text{ ◍ } (3 \cdot 10^3) + (6 \cdot 10^2) + (4 \cdot 10^1) + 5$

[**b**] $8943 \text{ ◍ } (8 \cdot 10^2) + (9 \cdot 10^1) + (4 \cdot 10^0) + 3$

[**c**] $53{,}762 \text{ ◍ } (5 \cdot 10^4) + (4 \cdot 10^3) + (7 \cdot 10^2) + (6 \cdot 10^1) + (2 \cdot 10^0)$

[**d**] $904{,}382 \text{ ◍ } (9 \cdot 10^4) + (4 \cdot 10^3) + (3 \cdot 10^2) + (8 \cdot 10^1) + 2$

[**e**] $67{,}028 \text{ ◍ } (6 \cdot 10^4) + (7 \cdot 10^3) + (1 \cdot 10^2) + (2 \cdot 10^1) + (8 \cdot 10^0)$

[**f**] $6{,}459{,}806 \text{ ◍ } (6 \cdot 10^6) + (4 \cdot 10^5) + (5 \cdot 10^4) + (9 \cdot 10^3) + (8 \cdot 10^2) + (6 \cdot 10^0)$

2. Give the correct symbol ($<$, $>$, or $=$) for each ◯.

[a] 68 ◯ 64
[b] 89 ◯ 98
[c] 349 ◯ 359
[d] 6758 ◯ 6578
[e] 5999 ◯ 6001
[f] 3989 ◯ 3899
[g] 49,753 ◯ 94,753
[h] 748,932 ◯ 784,923
[i] 6,004,400 ◯ 6,040,040

3. For each pair of numbers, write a true statement, using an inequality symbol and stating that the first number is either greater than or less than the second.

Example: For 29, 92, write $29 < 92$.

[a] 73, 75
[b] 39, 93
[c] 452, 442
[d] 3574, 3534
[e] 8878, 8888
[f] 6491, 6492
[g] 4672, 4592
[h] 1000, 999
[i] 0, 364

4. Write the correct symbol ($<$, $>$, or $=$) for each ◯. Try to do this without finding sums.

[a] $24 + 37$ ◯ $25 + 37$
[b] $356 + 428$ ◯ $355 + 429$
[c] $62 + 93$ ◯ $72 + 83$
[d] $249 + 356$ ◯ $239 + 346$
[e] $48 + 52$ ◯ $41 + 62$
[f] $342 + 76$ ◯ $442 + 66$
[g] $97 + 146$ ◯ $96 + 147$
[h] $218 + 812$ ◯ $128 + 821$
[i] $343 + 68$ ◯ $86 + 343$
[j] $487 + 352$ ◯ $497 + 342$
[k] $326 + 184$ ◯ $366 + 284$
[l] $3571 + 2856$ ◯ $3572 + 2854$
[m] $912 + 1461$ ◯ $1561 + 812$
[n] $6941 + 7213$ ◯ $6841 + 7513$

5. In each exercise, decide how much more the first sum is than the second.

[a] $86 + 79$, $76 + 79$
[b] $352 + 295$, $352 + 195$
[c] $342 + 564$, $242 + 464$
[d] $85 + 73$, $75 + 72$
[e] $86 + 49$, $76 + 39$
[f] $642 + 527$, $542 + 517$
[g] $642 + 527$, $542 + 537$
[h] $862 + 415$, $872 + 315$
[i] $845 + 271$, $745 + 261$
[j] $864 + 864$, $764 + 884$
[k] $6672 + 3594$, $5672 + 3694$
[l] $7143 + 4195$, $7243 + 2195$
[m] $444 + 222$, $333 + 333$
[n] $4444 + 3333$, $3333 + 2222$

6. Write the numeral for the number that is just one more than

[a] 19 [c] 399 [e] 91,099 [g] 9,999,999
[b] 99 [d] 4499 [f] 400,999 [h] 90,000,999

7. Write the numeral for the number that is just one less than

[a] 300 [c] 3900 [e] 30,000 [g] 5,000,500
[b] 4050 [d] 42,500 [f] 10,000,000 [h] 36,000,000

8. Write the numeral for each number indicated.

[a] 100 less than one million
[b] 1000 less than one million
[c] 100,000 less than one million
[d] 10,000 less than one million
[e] 101 less than one billion
[f] Half as much as one billion
[g] A half million less than 456,972,863
[h] The amount that must be added to 456,972,863 to give 457,000,000
[i] 52 million more than 846
[j] 8 billion more than 59
[k] A half million more than 6,587,346

★**9.** How old, to the nearest year, would a person be who had lived one million hours? Is he older or younger than a person who has lived one billion seconds?

Think

1. Check your pulse rate to find how many times your heart beats in one minute.

2. **Estimate** the answer to the questions. Then calculate the answers and compare them with your estimate.

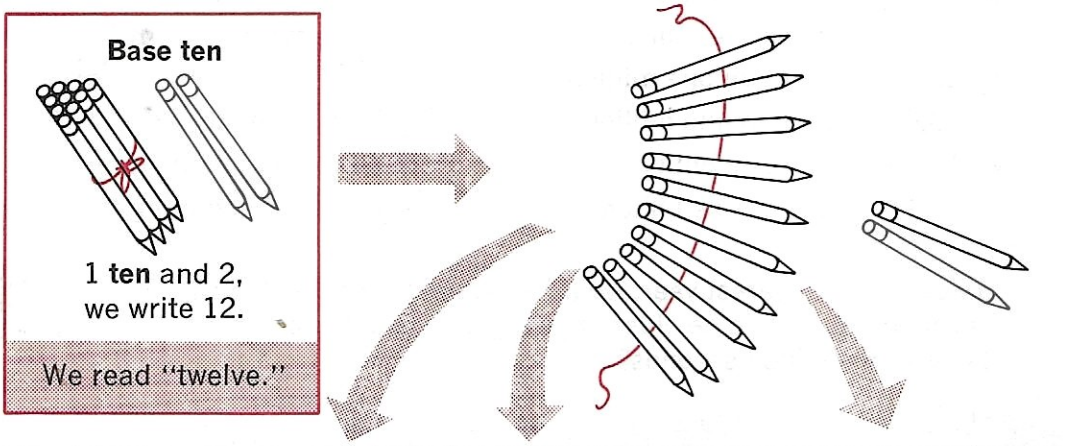
[a] If your heartbeat rate remains the same, how many days (to the nearest day) will it take for your heart to beat one million times?

[b] How many years (to the nearest year) will it take for your heart to beat one billion times?

1-7 OTHER NUMBER BASES

In our base ten numeration system, we group objects by tens and use place value to write numerals for numbers. This grouping by tens is probably done because man has ten fingers. If man had only four fingers on each hand, it might have been more natural to group by eights. It is an interesting game to use groupings other than ten and write other numerals for whole numbers.

Consider a group of 12 pencils. The different groupings of these pencils below show how to express the number 12 with different numerals.



Base eight	Base five	Base four
1 **eight** and 4, we write $14_{(8)}$.	2 **fives** and 2, we write $22_{(5)}$.	3 **fours** and 0, we write $30_{(4)}$.
We read "one, four, base 8."	We read "two, two, base 5."	We read "three, zero, base 4."

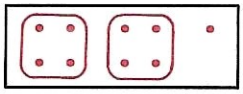
or or

Discussion Exercise

How would you write a base six, a base seven, and a base nine numeral to show 12 pencils?

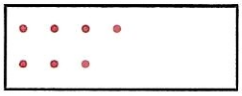
Exercises

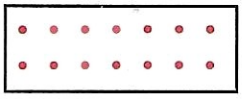
1. Think about grouping the following sets of dots by fours. Copy and complete each sentence.

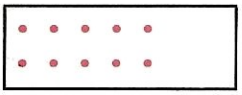
Example: 

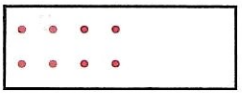
We think: 2 groups of four and 1

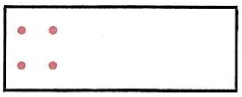
We write $21_{(4)}$

[a]  groups of four and . We write .

[b]  groups of four and . We write .

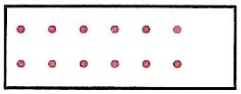
[c]  groups of four and . We write .

[d]  groups of four and . We write .

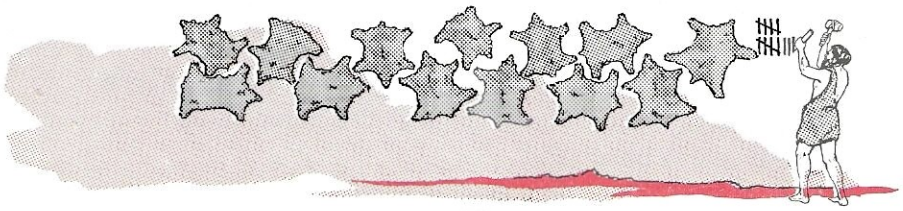
[e]  groups of four and . We write .

[f] groups of four and . We write .

[g] groups of four and . We write .

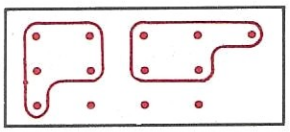
[h]  groups of four and . We write .

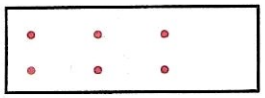
2. Cavemen may have kept hunting records with a base five numeration system. A certain caveman showed the following number of animal skins as his catch.

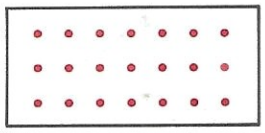


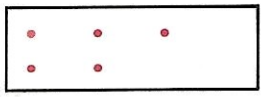
[a] Write the base ten numeral that tells how many skins in his catch.
[b] How many groups of five skins are in the catch?
[c] After all possible groups of five, how many extra skins?
[d] Write the base five numeral that tells how many skins in the catch.

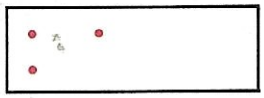
3. Think about grouping the following sets of dots as indicated by the given base. You may want to copy the dots and show the grouping. Write a numeral which indicates the number of dots in each box.

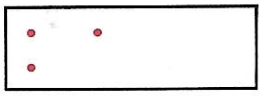
Example:  $23_{(5)}$

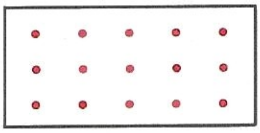
[a]  (4)

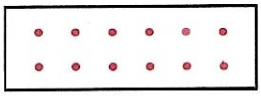
[b]  (10)

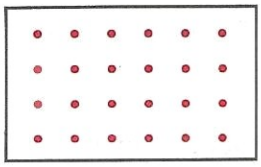
[c]  (8)

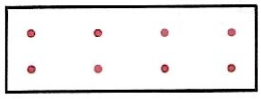
[d]  (2)

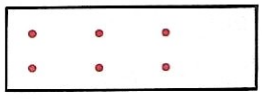
[e]  (5)

[f]  (6)

[g]  (4)

[h]  (9)

[i]  (8)

[j]  (3)

4. Give the missing base eight numerals.

$0_{(8)}$	$1_{(8)}$	$2_{(8)}$	$3_{(8)}$	a	b	c	$7_{(8)}$	$10_{(8)}$	$11_{(8)}$	d	e	f
	•	••	•••	••••	••••	••••	••••	••••	••••	••••	••••	••••

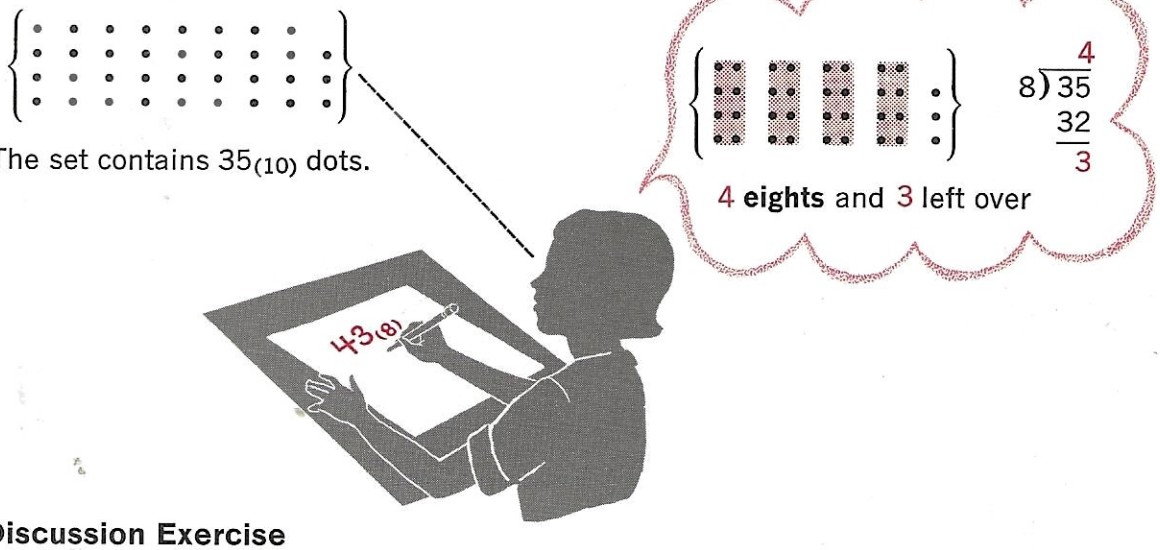
★**5.** "$312_{(4)}$" is a base four numeral that represents a certain whole number. Draw $312_{(4)}$ dots. (**Hint**: Think of the place-value scheme that we use in base ten.)

★**6.** When we use a base ten numeration system, we need only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to indicate the number of objects in any set. Tell how many digits we need in base

[a] four [b] five [c] eight

1-8 A BASE EIGHT NUMERATION SYSTEM

A **base eight** numeration system uses the eight digits 0, 1, 2, 3, 4, 5, 6, 7 and the idea of **place value** to write numerals for the whole numbers. It is interesting to compare the base eight (octal) system with our base ten (decimal) system. For example, if you know there are 35 dots in a set, how can you quickly find the base eight numeral that shows how many dots are in the set?



Discussion Exercise

If a set contains 58 dots, how can you quickly find a base eight numeral to represent this number?

Exercises

1. [a] How many eights in twenty-five? How many left over?
[b] Write the base eight numeral for $25_{(10)}$.
[c] How many eights in fifty-four? How many left over?
[d] Write the base eight numeral for $54_{(10)}$.
[e] How many eights in forty-eight? How many left over?
[f] Write the base eight numeral for $48_{(10)}$.

2. Copy and give the numeral for each ▒.

[a] $18_{(10)} = ▒_{(8)}$ [c] $32_{(10)} = ▒_{(8)}$ [e] $7_{(10)} = ▒_{(8)}$ [g] $52_{(10)} = ▒_{(8)}$
[b] $27_{(10)} = ▒_{(8)}$ [d] $38_{(10)} = ▒_{(8)}$ [f] $40_{(10)} = ▒_{(8)}$ [h] $63_{(10)} = ▒_{(8)}$

3. Study the example. Then write three numerals to complete each sentence.
Example: $21_{(8)}$ means 2 eights and 1, or 17 in base ten.

[a] $17_{(8)}$ means ▒ eights and ▒, or ▒ in base ten.
[b] $23_{(8)}$ means ▒ eights and ▒, or ▒ in base ten.
[c] $31_{(8)}$ means ▒ eights and ▒, or ▒ in base ten.
[d] $45_{(8)}$ means ▒ eights and ▒, or ▒ in base ten.
[e] $56_{(8)}$ means ▒ eights and ▒, or ▒ in base ten.

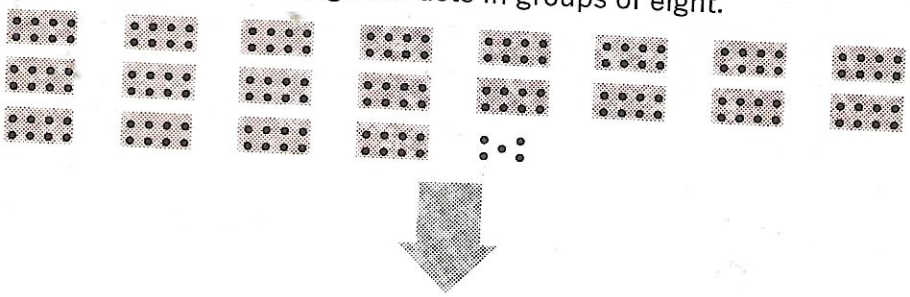
4. Solve the equations.

[a] $19_{(10)} = (2 \cdot 8) + n$
[b] $30_{(10)} = (n \cdot 8) + 6$
[c] $58_{(10)} = (n \cdot 8) + 2$
[d] $62_{(10)} = (7 \cdot 8) + n$
[e] $72_{(10)} = (n \cdot 8) + 0$
[f] $78_{(10)} = (9 \cdot 8) + n$

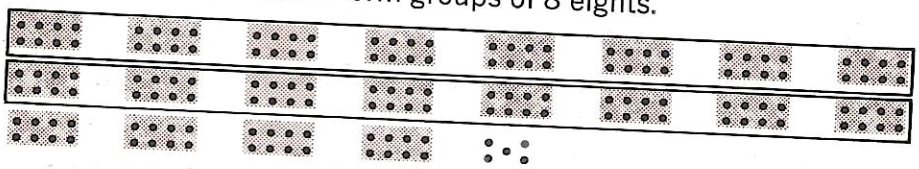
5. [a] What is the largest one-digit number in base eight notation?
[b] What is the largest two-digit number in base eight notation?

6. Study the example to see how to write a three-digit base eight numeral that tells how many dots are in a set of dots.

First arrange the dots in groups of eight.



Then form groups of 8 eights.



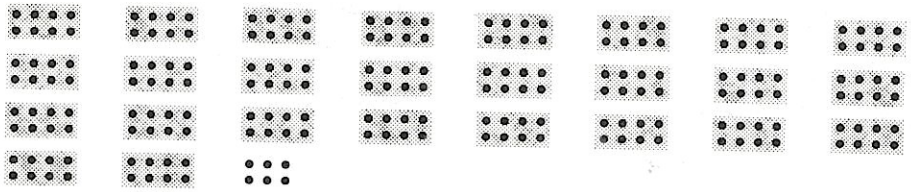
Since there are 2 groups of $8 \cdot 8$ (8 eights), 4 groups of 8, and 5 extra dots, the base eight numeral for the number of dots in this set is $245_{(8)}$.

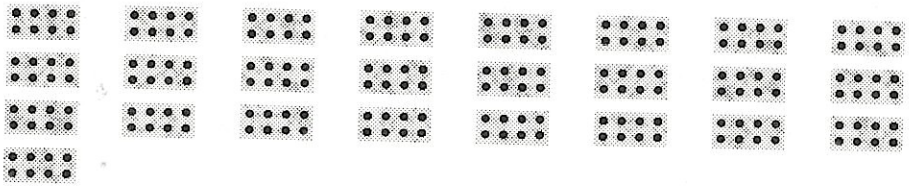
(A) $245_{(10)}$ means $(2 \cdot 10^2) + (4 \cdot 10^1) + 5$.

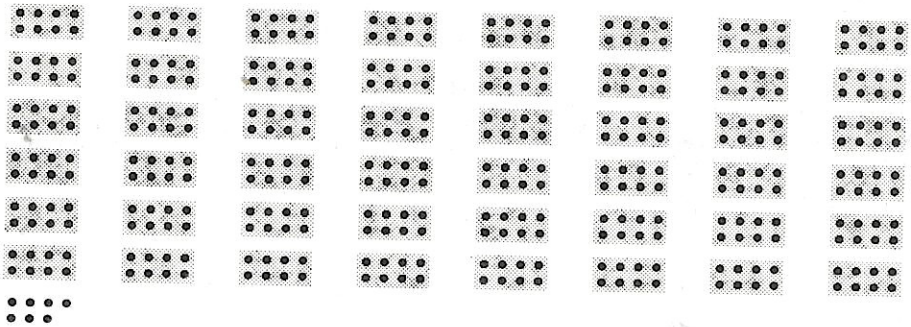
(B) $245_{(8)}$ means $(2 \cdot 8^2) + (4 \cdot 8^1) + 5$, or $165_{(10)}$.

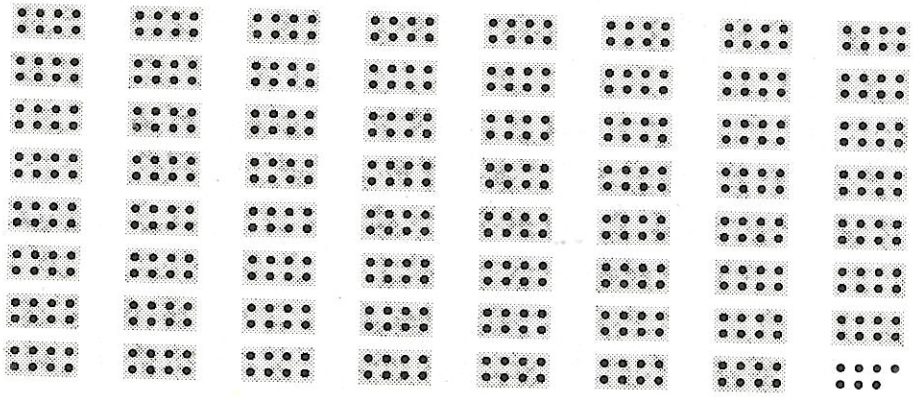
Note that just as 10^2 means $10 \cdot 10$, 8^2 means $8 \cdot 8$.

Give the base eight numeral for the number of dots in each set.

[a] 

[b] 

[c] 

[d] 

7. Copy each exercise and give the numeral for each ▒.

[a] $352_{(8)} = (▒ \cdot 8^2) + (▒ \cdot 8) + ▒$
[b] $352_{(8)} = ▒_{(10)}$
[c] $746_{(8)} = (▒ \cdot 8^2) + (▒ \cdot 8) + ▒$
[d] $746_{(8)} = ▒_{(10)}$
[e] $401_{(8)} = (▒ \cdot 8^2) + (▒ \cdot 8) + ▒$
[f] $401_{(8)} = ▒_{(10)}$
[g] $500_{(8)} = (▒ \cdot 8^2) + (▒ \cdot 8) + ▒$
[h] $500_{(8)} = ▒_{(10)}$

★ **8.** Think about place value in base ten and write the base eight numeral for a set that has one more dot than the set in Exercise 6**d**.

★ **9.** What number comes just after each of these numbers? Use base eight numerals.

[a] $77_{(8)}$ [b] $127_{(8)}$ [c] $777_{(8)}$ [d] $7077_{(8)}$ [e] $100777_{(8)}$

★ **10.** In base eight notation, $3412_{(8)}$ means $(3 \cdot 8^3) + (4 \cdot 8^2) + (1 \cdot 8) + 2$. Copy each exercise and give the missing numeral for each ▒.

[a] $3412_{(8)} = ▒_{(10)}$
[b] $5632_{(8)} = ▒_{(10)}$
[c] $4273_{(8)} = ▒_{(10)}$
[d] $2541_{(8)} = ▒_{(10)}$
[e] $7675_{(8)} = ▒_{(10)}$
[f] $7777_{(8)} = ▒_{(10)}$

Think

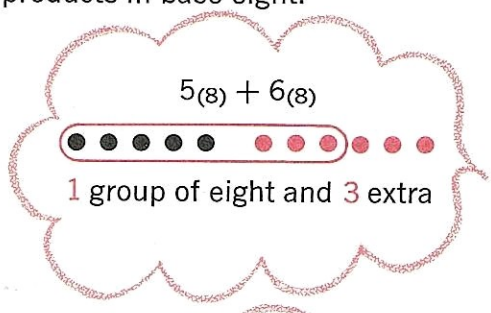
$8^1 = 8$
$8^2 = ▒$
$8^3 = ▒$
$8^4 = ▒$

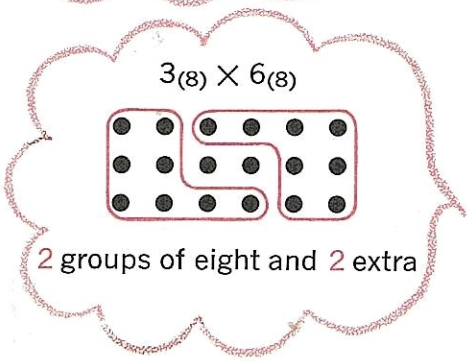
Copy and complete the table. Then use the table to find the correct digits for the letters. Then write the base eight numeral for the number shown in the colored screen.

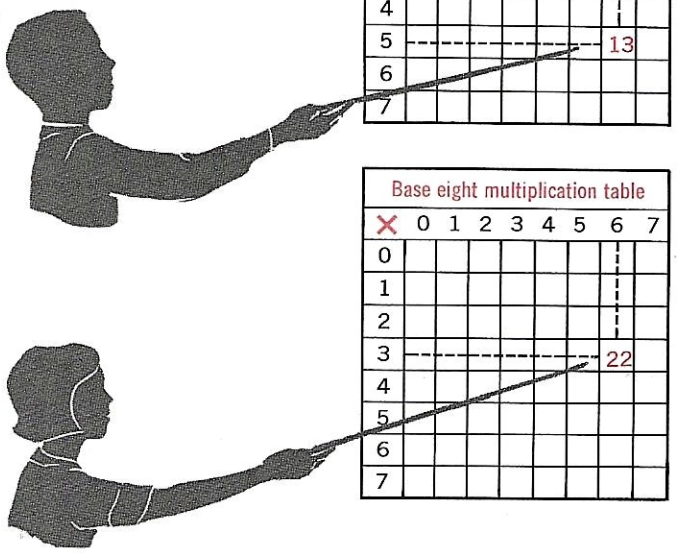
1. $235_{(10)} = (a \cdot 8^2) + (b \cdot 8^1) + c = ▒_{(8)}$
2. $669_{(10)} = (a \cdot 8^3) + (b \cdot 8^2) + (c \cdot 8^1) + d = ▒_{(8)}$
3. $8474_{(10)} = (a \cdot 8^4) + (b \cdot 8^3) + (c \cdot 8^2) + (d \cdot 8^1) + e = ▒_{(8)}$
4. $9999_{(10)} = (a \cdot 8^4) + (b \cdot 8^3) + (c \cdot 8^2) + (d \cdot 8^1) + e = ▒_{(8)}$

1-9 COMPUTATION IN BASE EIGHT

The tables will help you find sums and products in base eight.







Base eight addition table

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5							13	
6								
7								

Base eight multiplication table

×	0	1	2	3	4	5	6	7
0								
1								
2								
3							22	
4								
5								
6								
7								

Discussion Exercises

1. Explain how to find $7_{(8)} + 6_{(8)}$.

2. Explain how to find $4_{(8)} \cdot 7_{(8)}$.

Exercises

1. Copy and complete the base eight addition table above.

2. To help find these sums, use the addition table completed in Exercise 1.

[a] $15_{(8)} + 2_{(8)}$ [b] $15_{(8)} + 3_{(8)}$ [c] $16_{(8)} + 4_{(8)}$ [d] $26_{(8)} + 1_{(8)}$ [e] $26_{(8)} + 2_{(8)}$ [f] $26_{(8)} + 3_{(8)}$

[g] $24_{(8)} + 33_{(8)}$ [h] $24_{(8)} + 34_{(8)}$ [i] $24_{(8)} + 37_{(8)}$ [j] $54_{(8)} + 23_{(8)}$ ★[k] $44_{(8)} + 36_{(8)}$ ★[l] $54_{(8)} + 24_{(8)}$

3. Copy and complete the base eight multiplication table above.

4. To help find these products, use the multiplication table completed in Exercise 3.

[a] $12_{(8)} \times 3_{(8)}$ [b] $12_{(8)} \times 4_{(8)}$ [c] $12_{(8)} \times 5_{(8)}$ [d] $23_{(8)} \times 3_{(8)}$ [e] $34_{(8)} \times 2_{(8)}$ ★[f] $57_{(8)} \times 6_{(8)}$

★**5.** Find the differences. Use base eight numerals.

[a] $15_{(8)} - 7_{(8)}$ [b] $26_{(8)} - 7_{(8)}$ [c] $41_{(8)} - 5_{(8)}$ [d] $74_{(8)} - 47_{(8)}$

★**6.** Find the quotients. Use base eight numerals.

[a] $52_{(8)} \div 7_{(8)}$ [b] $61_{(8)} \div 7_{(8)}$ [c] $36_{(8)} \div 5_{(8)}$ [d] $556_{(8)} \div 6_{(8)}$

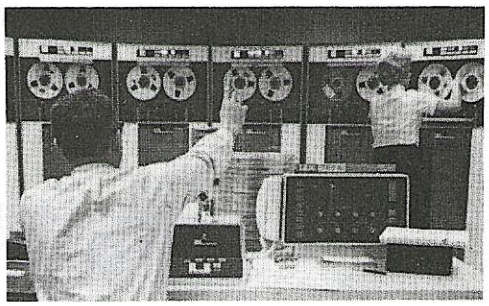
★**7.** Compute the following. Use base eight numerals.

[a] $645_{(8)} + 776_{(8)}$ [b] $732_{(8)} - 376_{(8)}$ [c] $537_{(8)} \times 46_{(8)}$ [d] $34_{(8)})\overline{2444_{(8)}}$

8. [a] Showing each base eight numeral, continue the doubling process until you reach $10000_{(8)}$. $1_{(8)}$, $2_{(8)}$, $4_{(8)}$, ▓, ▓, ▓, ▓, ...

[b] Is $10000_{(8)}$ greater or less than $5000_{(10)}$?

Think



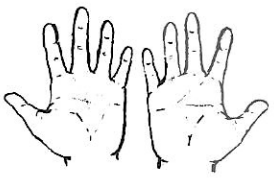
A computer control room. Making use of bases other than ten, computers can perform complicated calculations in thousandths or millionths of a second.

Numbo was in grade 22 and was 110 years old. He really felt quite young, however. He slept 101 hours each night, awoke at 21 o'clock, and ran 1212 yards in 22 seconds. About all the history he knew was that Columbus discovered America in the year 2001021.

1. The numbers in this story are represented using a number base other than ten. Find what number base is used and recopy the story, using the correct base ten numerals.
2. Write a story using a number base other than base ten. Give the story to a classmate to figure out.

1-10 OTHER NUMERALS

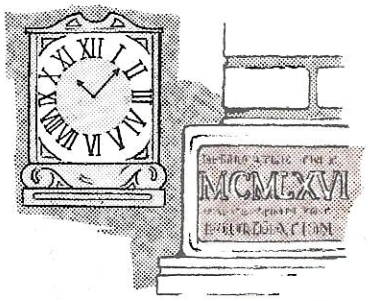
Since very early times, people have needed numbers. Throughout the ages **numbers have not changed, but many different numerals have been used to represent them.** Today we write the numeral 10 to represent the number of fingers on a man's two hands. The Romans and most people of medieval Europe wrote the numeral X. Egyptian scribes wrote the numeral ∩. In ancient Babylon the symbol ≺ was used. The Greek mathematicians wrote ι (the letter iota of their alphabet).



10, X, ∩, ≺, ι

Egyptian numerals 1700 B.C.		Our numerals
	(stroke) =	1
	(heel bone) =	10
	(coiled rope) =	100
	(lotus flower) =	1000
	(pointing finger) =	10,000
	(polliwog) =	100,000
	(astonished man) =	1,000,000

Babylonian numerals 1700 B.C.		Our numerals
	=	1
≺	=	10



Roman numerals 0 A.D.		Our numerals
I	=	1
V	=	5
X	=	10
L	=	50
C	=	100
D	=	500
M	=	1000

Exercises

1. The **Egyptian numeration system** did not use place value. To show larger numbers, they wrote symbols over again and **added** the value of each symbol. For example: ∩∩∩|| is a symbol for 132. Write each number, using Hindu-Arabic numerals.

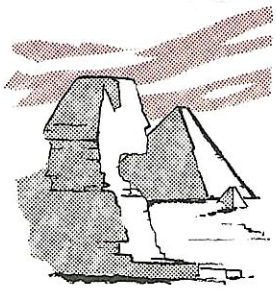
[a] ||| [b] ∩∩∩∩|| [c] ∩| [d] ∩∩|||||

[e] ∩∩∩ [f]

[g] | [h] ∩∩∩∩∩∩|||

2. Write each number, using Egyptian numerals.

[a] 18	[c] 426	[e] 1,000,010	[g] 1968	[i] 2,432,324
[b] 42	[d] 100,100	[f] 1,111,111	[h] 10,001	[j] 3,005,418



3. The meanings of the Egyptian numerals are not changed if the order of the symbols is changed. For example, 13 could be written as ∩||| or |∩|| or ||∩| or |||∩. Show the different ways that one hundred eleven could be written using Egyptian numerals.

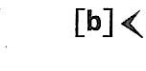
4. Write each sum, using Egyptian numerals.

[a] || + ∩∩ [b] ∩| + ∩| [c] 𓍢∩| + 𓍢𓍢∩∩||

5. The **Babylonian numeration system** used the idea of place value. It was a **base sixty** numeration system! The Babylonians used a piece of wood called a **stylus** to write in clay. They wrote numerals for numbers up to 59, using combinations of the two symbols shown in the Babylonian numeral table.

For example, is a symbol for 6 and is a symbol for 44.

Write each number, using Hindu-Arabic numerals.

[a] [b]  [c] [d]

6. Write each number, using Babylonian numerals.

[a] 7 [b] 16 [c] 36 [d] 40 [e] 55 [f] 11

★**7.** To write numerals for numbers over 59, the Babylonians used the same two symbols over again. By using an alternating pattern, they could show place value. At first, they had no symbol for zero, and it was difficult to decide what they meant. Later, the mark 𒌋 represented an empty space. Notice that it may not always be clear what number the Babylonian symbol represents.

Ⓐ was a symbol for $(1 \cdot 60) + 0$, or 60.

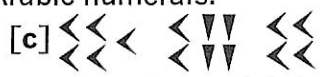
Ⓑ was a symbol for $(21 \cdot 60^2) + (0 \cdot 60) + 2$, or 75,602

Ⓒ was a symbol for $(42 \cdot 60^2) + (35 \cdot 60) + 23$, or 153,323

Write each number, using Hindu-Arabic numerals.

[a]

[b]

[c] 

[d]

8. The **Roman numeration system** does not use place value. When two Roman symbols are placed side by side, we add or subtract to determine the number represented. We **add** when the smaller number is written on the right.

XI means $10 + 1$

We **subtract** when the smaller number is written on the left.

IX means $10 - 1$

The seven basic Roman numerals	
I = 1	C = 100
V = 5	D = 500
X = 10	M = 1000
L = 50	

Copy and complete the following. Each numeral is an "addition numeral."

[a] VI = 6
[b] LI = ▒
[c] LX = ▒
[d] XV = ▒
[e] XVI = ▒
[f] CL = ▒
[g] CX = ▒
[h] CV = ▒
[i] DC = ▒
[j] DXI = ▒
[k] MD = ▒
[l] MV = ▒

9. Copy and complete the following. The six basic "subtraction numerals" are shown here.

[a] IV = 4
[b] XL = ▒
[c] CD = ▒
[d] IX = ▒
[e] XC = ▒
[f] CM = ▒

10. Write a Roman numeral for each Hindu-Arabic numeral.

Examples: 7 = VII, 24 = XXIV

[a] 3
[b] 6
[c] 11
[d] 34
[e] 140
[f] 529
[g] 4
[h] 19
[i] 26
[j] 38
[k] 126
[l] 342

11. Write a Hindu-Arabic numeral for each Roman numeral.

[a] CCLXII
[b] DXLV
[c] CDXLIV
[d] CDXXXVII
[e] CMLIX
[f] MMLI
[g] MDCCLXXVI
[h] MCMLXVIII

12. Give the Roman numerals for the numbers indicated.

[a] 1 through 10
[b] 15 through 20
[c] 40 through 50
[d] 120 through 130
[e] 395 through 400
[f] 990 through 1000

13. When the Romans placed a bar above one of the six numerals V, X, L, C, D, or M, the resulting numeral represented a number 1000 times as large.

Examples: $\overline{V} = 5000$, $\overline{C} = 100{,}000$

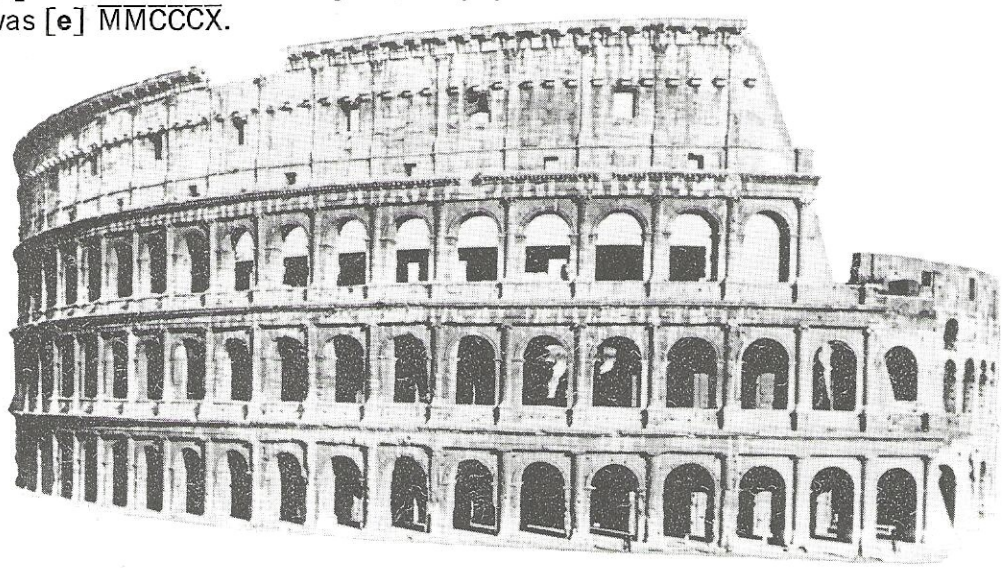
Copy each exercise and give the missing Hindu-Arabic numerals.

[a] $\overline{X} =$ ▒▒ [b] $\overline{L} =$ ▒▒ [c] $\overline{D} =$ ▒▒ [d] $\overline{M} =$ ▒▒

★**14.** Early Romans did not use "subtraction numerals." Only later were the "subtraction numerals" I, X, and C used as follows: to the left of an X or V, we write only an I. To the left of an L or C, we write only an X. To the left of an M or D, we write only a C. Write all the "two letter" numerals that can be formed using only the seven basic Roman numerals and give the Hindu-Arabic numeral for each.

★**15.** Write the Roman numeral for the year 1974.

★**16.** Write the Hindu-Arabic numeral for each Roman numeral in this paragraph: Rome was first settled around [a] DCCLIII B.C. One of the best known of ancient buildings in Rome is the large, half-ruined Colosseum, built around [b] LXXV A.D. It held about [c] $\overline{X}L\overline{V}$ spectators. The catacombs, underground chambers where Christians were buried, take up a total area of about [d] DC acres. In a recent year, the population of Rome was [e] $\overline{MM}\overline{CCC}X$.



The Colosseum in Rome

★**17.** Add each of the following, without changing to Hindu-Arabic numerals.

[a] CXI + XVI [b] CM + D [c] LXVII + CLXVI

★**18.** Subtract each of the following, without changing to Hindu-Arabic numerals.

[a] XXXIII − XII [b] MDCCXXXVIII − MCCXXI

★**19.** Find the product of CLVI and XV, without changing to Hindu-Arabic numerals.

Think

In the Chinese-Japanese system, the numerals are written vertically. A multiplicative grouping system is used. Study the numerals given for 30, 38, and 456.

30	38	456
↓	↓	↓
三 十	三 十 八	四 百 五 十 六

Write Chinese-Japanese numerals for

1. 20 **3.** 99 **5.** 5167

2. 47 **4.** 638 **6.** 9034

Chinese-Japanese numerals		Our numerals
一	=	1
二	=	2
三	=	3
四	=	4
五	=	5
六	=	6
七	=	7
八	=	8
九	=	9
十	=	10
百	=	100
千	=	1000

CHAPTER REVIEW

Key Concepts

1. A **number** is an idea. A **numeral** is a symbol used to represent a number.
2. Throughout history, men have developed **numeration systems** for writing numerals for numbers. Our system is based on grouping by tens (base ten) and uses **place value**. This idea enables us to write numerals for all the whole numbers, using only the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
3. **Exponents** can simplify the writing of larger numbers.
4. It is often desirable to give an **approximation** for a large number. We can round a number to any desired place.
5. **Place value** helps us decide which of two numbers is greater.
6. It is possible to group by numbers other than ten. In base eight, for example, there are fewer digits and fewer basic multiplication facts to learn than in base ten. Computation in base eight depends upon the same basic grouping principles as computation in base ten.
7. Hindu-Arabic numerals are relatively modern. Egyptians, Babylonians, and Romans used different numerals to represent numbers.

Vocabulary List

abacus
digit
empty set
expanded notation
exponent
factor
Hindu-Arabic numeral
inequality symbol
number
number base
numeral
numeration system
period
place value
power
scientific notation
set
whole number

Review Exercises

1. Write a digit that shows how many objects are in each set.
   [a] The set of letters used in the word "number"
   [b] The set of toes on a foot
   [c] The set of Atlas missiles in your classroom

2. Which would you write on paper, a **number** or a **numeral**?

3. Write 6527, using expanded notation.

4. In the numeral **563,412,389**, what digit is in each of these places?
   [a] Ones
   [b] Thousands
   [c] Millions
   [d] Tens
   [e] Ten thousands
   [f] Ten millions
   [g] Hundreds
   [h] Hundred thousands
   [i] Hundred millions

5. Which expression is not a numeral for 10?
   [a] $10 \cdot 1$
   [b] $10^2 \div 10$
   [c] $10 + 0$
   [d] $10 + 1$
   [e] 10^1

6. Write the usual Hindu-Arabic numeral for $(9 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (8 \cdot 10 \cdot 10 \cdot 10) + (7 \cdot 10 \cdot 10) + (6 \cdot 10) + 5$.

7. [a] 10^4 means 10 used as a factor _?_ times.
   [b] $10^4 =$ ▒

8. Write 100,000, using 10 with an exponent.

9. Write the usual Hindu-Arabic numeral for $(4 \cdot 10^4) + (7 \cdot 10^2) + (6 \cdot 10^1) + (9 \cdot 10^0)$.

10. Write 7648, using expanded notation (exponent form).

11. Use scientific notation to write the number in the following statement: The average diameter of the planet Uranus is 30,000 mi.

12. Give the product **P** and the missing power of ten.

$$10^2 \cdot 10^{\square} = \square$$
$$\downarrow \quad\quad \downarrow \quad\quad \uparrow$$
$$100 \cdot 1000 = \mathbf{P}$$

13. Round the number 3,856,467 to the nearest
- [a] hundred
- [b] thousand
- [c] hundred thousand
- [d] million

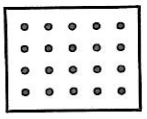
14. Give the correct symbol ($<$, $>$, or $=$) for each $\bigcirc$.

[a] $347 \bigcirc (3 \cdot 10^2) + (5 \cdot 10^1) + 7$

[b] $637 + 246 \bigcirc 627 + 256$

15. Write the number that is one less than one million.

16. [a] Write the base ten numeral that tells how many dots.



[b] Write the base eight numeral that tells how many dots.

[c] Write the base five numeral that tells how many dots.

17. [a] How many eights in 34?

[b] How many left over?

[c] Write the base eight numeral for $34_{(10)}$.

18. [a] $5_{(8)} + 5_{(8)} = \square_{(8)}$

[b] $4_{(8)} \cdot 3_{(8)} = \square_{(8)}$

19. Use scientific notation to represent the weight (20,000 lb) of a large elephant.

20. Write the Egyptian, Roman, and Babylonian numerals for 124.

CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 8–13.

BERGAMINI, DAVID, and the Editors of *Life. Mathematics.* Life Science Library; New York, Time, Inc., 1963; pp. 8–18.

DAVIS, PHILIP. *Lore of Large Numbers,* New Mathematics Library; Syracuse, New York, Singer Company, 1961; Part I.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Computing Devices.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1961; pp. 37–52.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Invitation to Mathematics. Ibid.,* 1960, pp. 1–6.

———. *Understanding Numeration Systems. Ibid.,* entire pamphlet.

KENYON, RAYMOND. *I Can Learn About Calculators and Computers.* Evanston, Illinois, Harper and Row, 1961; pp. 1–24.

SMITH, DAVID E. *History of Mathematics.* New York, Dover Publications, 1953; Vol. II, pp. 36–88.

Activities

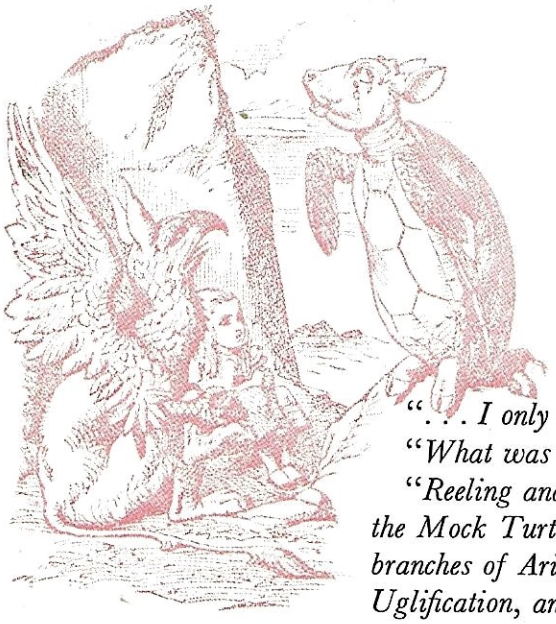
A. Read and report about the abacus. There are several different kinds. What people have used the abacus? What people still use the abacus today? Construct an abacus of your own. (See *I Can Learn About Calculators and Computers,* pp. 44–52.)

B. Write briefly about ten different types of jobs requiring a knowledge of mathematics. (See *Invitation to Mathematics.*)

C. Find discussions of the computing devices built by Pascal, Leibniz, Babbage, and Kelvin. Choose one of these and briefly describe its operation.

D. Write about one use of computers in space exploration.

E. Find the names of the next five periods beyond the sextillions. (See *Lore of Large Numbers.*)



". . . I only took the regular course."

"What was that?" inquired Alice.

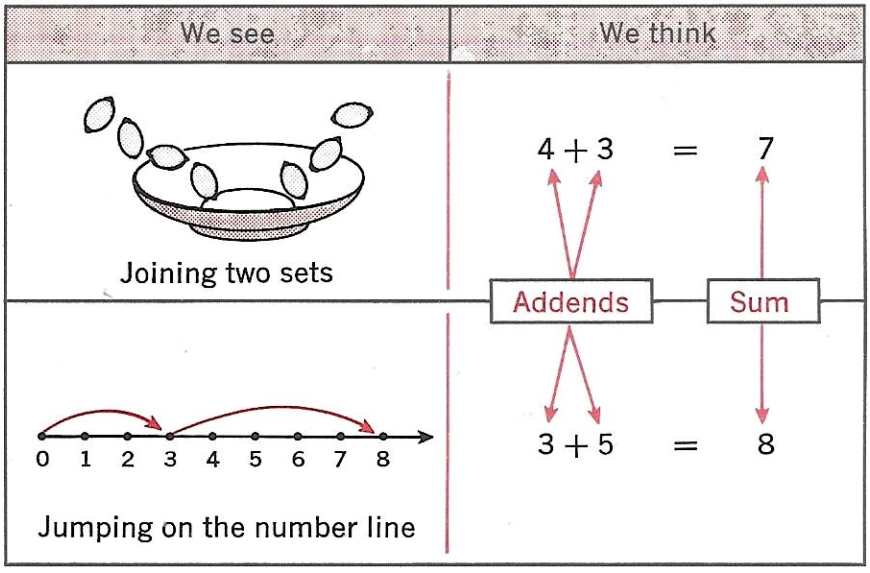
"Reeling and Writhing, of course, to begin with," the Mock Turtle replied; "and then the different branches of Arithmetic—Ambition, Distraction, Uglification, and Derision."

2

Whole Number Operations

2-1 ADDITION AND SUBTRACTION

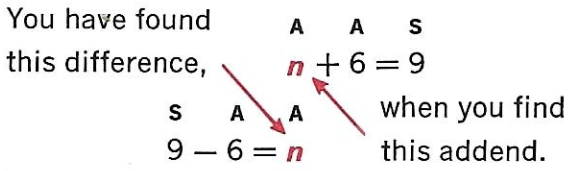
Addition is an operation that combines a first number and a second number to give one number. We call the first two numbers **addends**. We call the result the **sum** of the two addends. The table shows two ways we might think about addition.



Subtraction can best be thought of in terms of addition. We can think of subtraction as "undoing" addition.

	Do	Undo	
Start	Add 3	Subtract 3	End
5	$\longrightarrow 5+3 \longrightarrow$	$8-3 \longrightarrow$	5
2	$\longrightarrow 2+3 \longrightarrow$	$5-3 \longrightarrow$	2

Subtraction is an operation that combines a first number and a second number to give one number. The first number is the sum of the two addends. The third number is sometimes called the **difference** of the first two. In the diagram below, the A and S are to help you think about addends and sums.



The relationship demonstrated by these examples is called the **inverse** relation between addition and subtraction.

Exercises

1. Solve the equations. (To solve an equation like $n + 4 = 7$, we write $n = 3$.) Example: $6 + 5 = a$, $a = 11$

[a] $4 + 5 = a$	[g] $8 + 4 = m$	[m] $y + 0 = 8$	[s] $6 + 6 = n$
[b] $9 + n = 10$	[h] $d + 9 = 17$	[n] $6 + 9 = q$	[t] $9 + d = 12$
[c] $b + 6 = 15$	[i] $0 + r = 13$	[o] $5 + f = 5$	[u] $r + 3 = 11$
[d] $4 + a = 11$	[j] $7 + 7 = s$	[p] $n + 8 = 15$	[v] $7 + 6 = s$
[e] $9 + 8 = n$	[k] $9 + t = 16$	[q] $c + 7 = 13$	[w] $t + 9 = 13$
[f] $6 + c = 12$	[l] $x + 8 = 12$	[r] $7 + a = 10$	[x] $9 + h = 18$

2. Think about finding the missing addend. Then solve each subtraction equation.

[a] $? + 6 = 13$
$13 - 6 = r$

[b] $? + 5 = 11$
$11 - 5 = a$

[c] $? + 7 = 16$
$16 - 7 = n$

[d] $? + 4 = 12$
$12 - 4 = x$

[e] $? + 0 = 8$
$8 - 0 = c$

[f] $? + 7 = 14$
$14 - 7 = t$

[g] $? + 9 = 10$
$10 - 9 = q$

[h] $? + 1 = 7$
$7 - 1 = m$

[i] $? + 9 = 16$
$16 - 9 = s$

[j] $? + 8 = 13$
$13 - 8 = b$

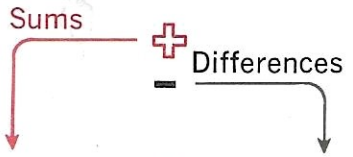
[k] $? + 11 = 11$
$11 - 11 = d$

[l] $? + 8 = 17$
$17 - 8 = y$

3. Solve the equations.

[a] $4 + a = 12$
[b] $x + 9 = 18$
[c] $10 - 6 = a$
[d] $n + 8 = 14$
[e] $11 - r = 9$
[f] $16 - 0 = d$
[g] $q - 6 = 6$
[h] $5 + s = 11$
[i] $8 + 7 = t$
[j] $13 - n = 8$
[k] $p + 7 = 15$
[l] $12 - a = 3$
[m] $b - 6 = 9$
[n] $7 + 7 = c$
[o] $10 - x = 2$
[p] $y + 13 = 13$
[q] $14 - 8 = n$
[r] $12 - s = 0$

4. Find the numbers a and b in each exercise. The answer for Part a is $a = 7$ and $b = 2$, since $9 = 7 + 2$ and $7 - 2 = 5$.



[a]	9	a	+ −	b	5
[b]	12	a	+ −	b	6
[c]	14	a	+ −	b	2
[d]	14	a	+ −	b	0
[e]	17	a	+ −	b	7
[f]	19	a	+ −	b	3
[g]	14	a	+ −	b	14
[h]	17	a	+ −	b	15

5. The examples illustrate how we use parentheses. Solve the equations.

A $(9 - 3) + 2 =$

$6 + 2 = 8$

B $13 - (6 + 2) =$

$13 - 8 = 5$

[a] $(14 - 9) + 7 = x$

[b] $9 + (10 - 2) = n$

[c] $(12 + 4) - 9 = a$

[d] $16 - (8 + 1) = c$

[e] $10 + (6 - 2) = q$

[f] $(17 - 8) + 4 = b$

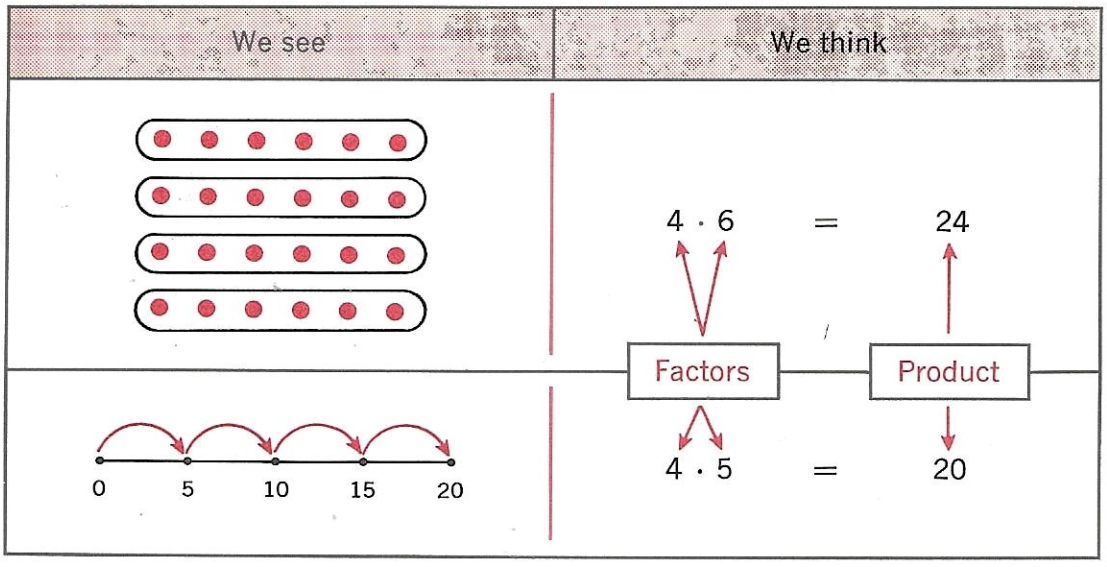
Think

In a group of 24 children, 12 wear glasses.

1. If you know that none of the boys wear glasses, what is the greatest number of boys possible in the group?
2. If there are 9 boys in the group, what is the fewest possible number of girls who wear glasses? What is the greatest possible number?
3. Now suppose there are 2 boys for every girl and no girl wears glasses. How many boys do not wear glasses?

2-2 MULTIPLICATION AND DIVISION

Multiplication is an operation that combines a first number and a second number to give one number. We call the first two numbers **factors**. We call the result the **product** of the two factors. The table shows two ways we might think about multiplication.



Division can best be thought of in terms of multiplication. We can think of division as "undoing" multiplication.

	Do		Undo	
Start	Multiply by 3		Divide by 3	End
5	$\rightarrow$ $5 \cdot 3$	$\rightarrow$	$15 \div 3$	$\rightarrow$ 5
2	$\rightarrow$ $2 \cdot 3$	$\rightarrow$	$6 \div 3$	$\rightarrow$ 2

Division is an operation that combines a first number and a second number to give one number. The first number is the product of the two factors. The third number is sometimes called the **quotient** of the first two. In the diagram below, the F and P are to help you think about factors and products.

You have found this quotient,

F F P
$n \cdot 4 = 28$

P F F
$28 \div 4 = n$

when you find this factor.

The relationship demonstrated by the diagram is called the **inverse** relation between multiplication and division.

Exercises

1. Solve the equations.

[a] $2 \cdot 6 = a$
[b] $8 \cdot n = 40$
[c] $9 \cdot 3 = b$
[d] $c \cdot 4 = 28$
[e] $6 \cdot d = 36$
[f] $x \cdot 8 = 56$
[g] $7 \cdot y = 21$
[h] $3 \cdot 5 = r$
[i] $8 \cdot s = 16$
[j] $t \cdot 4 = 36$
[k] $8 \cdot 1 = p$
[l] $7 \cdot m = 49$
[m] $4 \cdot 8 = y$
[n] $a \cdot 6 = 48$
[o] $2 \cdot 9 = n$
[p] $5 \cdot x = 35$
[q] $r \cdot 6 = 12$
[r] $5 \cdot a = 45$
[s] $7 \cdot 4 = g$
[t] $n \cdot 3 = 24$
[u] $0 \cdot 8 = q$
[v] $7 \cdot s = 42$
[w] $h \cdot 6 = 54$
[x] $9 \cdot 3 = f$

2. Think about finding the missing factor. Then solve each division equation.

[a] $? \cdot 4 = 24$
$24 \div 4 = n$

[b] $? \cdot 7 = 28$
$28 \div 7 = a$

[c] $? \cdot 5 = 30$
$30 \div 5 = r$

[d] $? \cdot 3 = 3$
$3 \div 3 = t$

[e] $? \cdot 6 = 42$
$42 \div 6 = q$

[f] $? \cdot 8 = 40$
$40 \div 8 = s$

[g] $? \cdot 2 = 18$
$18 \div 2 = h$

[h] $? \cdot 9 = 0$
$0 \div 9 = b$

[i] $? \cdot 5 = 35$
$35 \div 5 = d$

[j] $? \cdot 3 = 27$
$27 \div 3 = c$

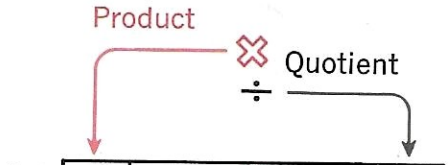
[k] $? \cdot 6 = 48$
$48 \div 6 = m$

[l] $? \cdot 7 = 7$
$7 \div 7 = d$

3. Solve the equations.

[a] $6 + a = 13$
[b] $f - 8 = 6$
[c] $5 \cdot j = 35$
[d] $14 \div r = 7$
[e] $k \cdot 4 = 36$
[f] $s \div 4 = 8$
[g] $b + 9 = 11$
[h] $6 \cdot 8 = f$
[i] $13 - g = 8$
[j] $m \cdot 6 = 36$
[k] $t \div 8 = 5$
[l] $7 + 7 = c$
[m] $7 \cdot 7 = n$
[n] $7 \div 7 = u$
[o] $7 - 7 = h$
[p] $v \div 6 = 8$
[q] $7 \cdot p = 35$
[r] $d + 8 = 17$
[s] $14 - 5 = i$
[t] $7 \cdot 9 = b$
[u] $8 \div w = 1$
[v] $q \cdot 6 = 0$
[w] $6 + e = 16$
[x] $x \div 9 = 1$

4. Find the numbers a and b in each exercise. The answer for Part a is $a = 6$ and $b = 3$, since $18 = 6 \times 3$ and $6 \div 3 = 2$.



[a]	18	a	⊠ ÷	b	2
[b]	12	a	⊠ ÷	b	3
[c]	27	a	⊠ ÷	b	3
[d]	32	a	⊠ ÷	b	2
[e]	20	a	⊠ ÷	b	5
[f]	16	a	⊠ ÷	b	1
[g]	49	a	⊠ ÷	b	1
[h]	25	a	⊠ ÷	b	1

5. Solve the equations. Remember, the parentheses tell you which operation to do first.

[a] $(3 \cdot 6) \div 2 = n$
[b] $5 + (8 - 3) = a$
[c] $4 \cdot (7 + 2) = c$
[d] $(21 \div 3) + 6 = x$
[e] $18 - (6 \cdot 3) = b$
[f] $(24 \div 6) \cdot 8 = m$
[g] $6 \cdot (6 \div 6) = v$
[h] $6 \div (6 \div 6) = s$
[i] $(6 \cdot 6) \div 6 = v$
[j] $6 + (6 \div 6) = y$
[k] $(6 + 6) \div 6 = a$
[l] $6 - (6 \div 6) = n$

★**6.** Solve the equations.

[a] $(2 \cdot x) + 3 = 13$
[b] $(3 \cdot n) + 4 = 10$
[c] $7 + (2 \cdot n) = 27$
[d] $10 - (5 \cdot a) = 0$
[e] $(2 \cdot b) + (2 \cdot b) = 20$
[f] $(3 \cdot d) + (2 \cdot d) = 35$
[g] $(6 \cdot n) - 3 = 27$
[h] $(6 \cdot r) + 3 = 33$
[i] $5 \cdot (s + 3) = 25$
[j] $5 \cdot (t - 3) = 25$
[k] $30 = (2 \cdot a) + 10$
[l] $(3 \cdot x) = 7 + (2 \cdot x)$
[m] $17 = (10 \cdot c) - 3$
[n] $(7 \cdot m) + 7 = 7$
[o] $(7 \cdot q) + 7 = 14$
[p] $(7 \cdot s) - 7 = 0$
[q] $(5 \cdot b) = 10 + (3 \cdot b)$
[r] $(8 \cdot c) + 10 = 34$

Think

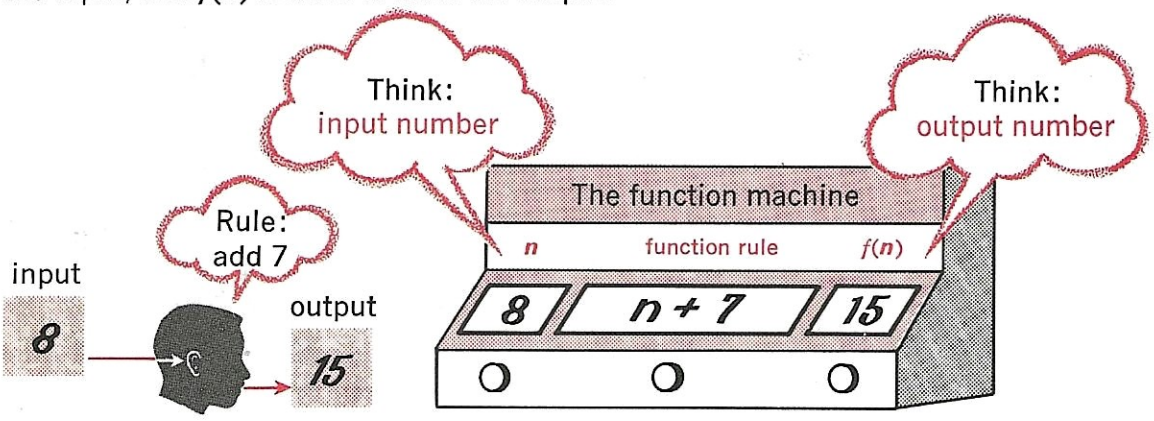
MENU

Beverages	Milk 15¢ Coffee 10¢
Sandwiches	Cheese 30¢ Ham 15¢ Hot Dog 25¢ Tuna 35¢
Soups	Bean 20¢ Vegetable 30¢ Chili 45¢

1. Jane decided to have a beverage and a sandwich for lunch. How many different lunches could she order?
2. Joe decided to have a beverage, a sandwich, and soup. How many different lunches could he get?
3. Pat wanted only soup and a sandwich. How many different lunches could she order?

2-3 OPERATIONS AND THE FUNCTION MACHINE

You can best understand the function machine by comparing the two pictures below. On the function machine, n is used to label the input, and $f(n)$ is used to label the output.



The function rule on the function machine tells how to find the output number $f(n)$ for a given input number n. For the example above, we describe the function rule as

$$\text{output number} = \text{input number} + 7$$

and write

$$f(n) = n + 7$$

Discussion Exercises

1. Jean and Sue were playing a "guess-the-function-rule game." See if you can guess the rule for this game.

When Jean said 3, Sue said 7.
When Jean said 5, Sue said 11.
When Jean said 7, Sue said 15.
When Jean said 4, Sue said 9.
When Jean said 2, Sue said 5.
When Jean said 0, Sue said 1.

2. Explain why you get the same result for both function rules.

(A) $f(n) = 3 \cdot n$ (B) $f(n) = n + n + n$

3. In Exercise 2, you can see how repeatedly adding the same number is an operation related to multiplication. Explain how subtraction and division are related in a similar way.

4. Give the output numbers for each gray space in the table.

5. Make up a function rule using addition and subtraction that will give the same result as the rule in Exercise 4.

	function rule $(n \cdot 3) \div 3$	
	input n	output $f(n)$
[a]	5	
[b]	2	
[c]	7	

Exercises

1. Using the function rule $f(n) = n \cdot 6$, give the output number for each of the following input numbers.

[a] 5 [b] 3 [c] 7 [d] 1 [e] 10 [f] 2 [g] 0 [h] 100

In Exercises 2 through 7, give the missing numbers for each table.

2.

	function rule $(n \cdot 4) \div 2$	
	input n	output $f(n)$
	2	4
	3	6
[a]	4	
[b]	1	
[c]	6	

3.

	function rule $(n + 6) - 2$	
	input n	output $f(n)$
	5	9
[a]	3	
[b]	0	
[c]	6	
[d]	9	

4.

	function rule $(n + 8) - 7$	
	input n	output $f(n)$
[a]	5	
[b]	7	
[c]	0	
[d]	6	
[e]	1	

5.

	function rule $n \cdot 3$	
	n	$f(n)$
[a]	6	
[b]	5	
[c]		12
[d]		0
[e]		21

6.

	function rule $n \div 3$	
	n	$f(n)$
[a]	15	
[b]		4
[c]		8
[d]	0	
[e]		10

7.

	function rule $(n + 6) - 3$	
	n	$f(n)$
[a]		10
[b]	6	
[c]		3
[d]		20
[e]	5	

For Exercises 8 through 16, study each table and invent a suitable function rule.

8. function rule

n	$f(n)$
6	10
1	5
4	8
0	4
10	14

9. function rule

n	$f(n)$
3	6
5	10
0	0
8	16
10	20

10. function rule

n	$f(n)$
3	5
8	15
5	9
6	11
2	3

11. function rule

n	$f(n)$
2	4
3	9
5	25
0	0
8	64

12. function rule

n	$f(n)$
4	14
2	8
6	20
0	2
1	5

13. function rule

n	$f(n)$
8	5
10	6
4	3
14	8
20	11

14. function rule

n	$f(n)$
6	6
9	9
14	14
0	0
34	34

★15. function rule

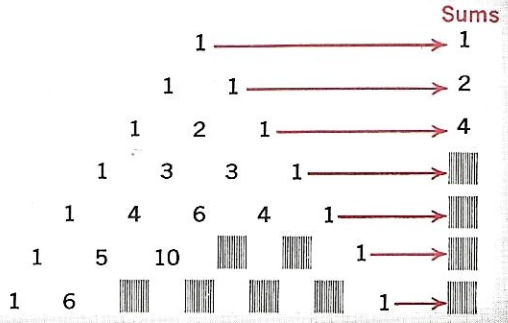
n	$f(n)$
2	6
5	30
1	2
7	56
4	20

★16. function rule

n	$f(n)$
12	14
17	24
10	10
6	2
18	26

Think

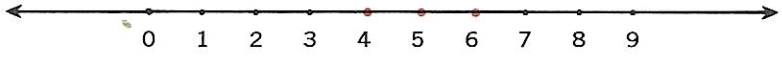
Copy the triangular array of numbers and try to give the missing numbers. Find the sum of the numbers in each row. Do you see a pattern? Now find the next row.



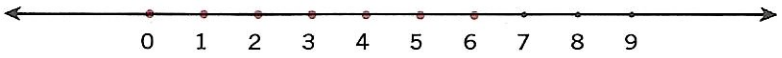
2-4 GRAPHING

In this section, you will learn to graph sets of numbers and sets of number pairs. You can graph sets of numbers on a number line.

A Graph the set of numbers {4, 5, 6}.

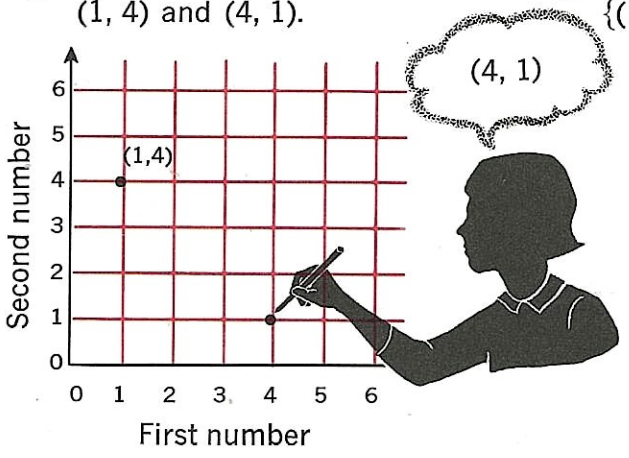


B Graph the set of all whole numbers less than 7.

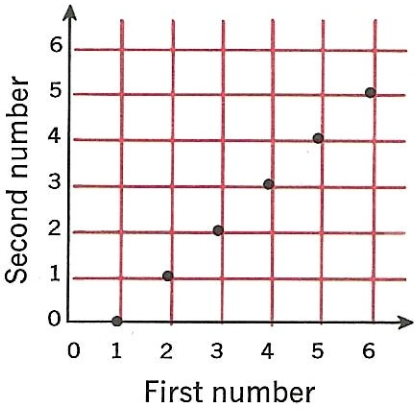


To graph sets of number **pairs**, we need two number lines.

C Graph the number pairs (1, 4) and (4, 1).



D Graph the set of number pairs {(1, 0), (2, 1), (3, 2), (4, 3), (5, 4), (6, 5)}.



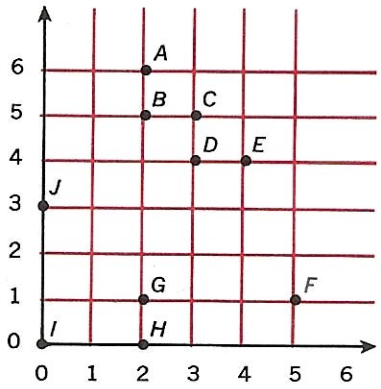
Exercises

1. Draw a six-inch line. Mark points at every half inch. Label your points from 0 to 12. Now, graph (by marking in heavy pencil or in a second color) the set $\{2, 3, 4, 5, 6\}$.

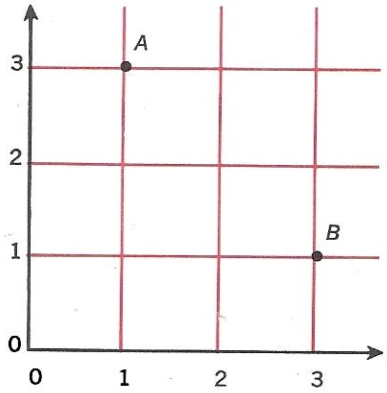
2. Draw a different number line for each exercise. Graph each set of numbers.

[a] $\{1, 3, 5, 7\}$ [b] $\{0, 1, 2, 3, 4\}$ [c] All whole numbers between 2 and 10
[d] All whole numbers greater than 0 and less than 6 [e] All whole numbers less than 5

3. Give the number pair for each point.
The number pair for point *A* is (2, 6).
The number pair for point *B* is (2, 5).

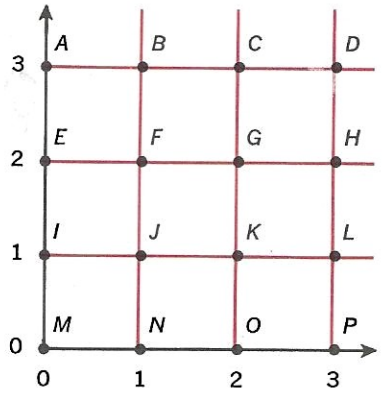


4. Which point is the graph of the number pair (3, 1)? Of the pair (1, 3)? How do you know?



5. Each point on this graph is labeled with a letter. Give the correct letter for each number pair.

[a] (1, 3)
[b] (3, 1)
[c] (3, 2)
[d] (2, 3)
[e] (2, 2)
[f] (0, 2)
[g] (2, 0)
[h] (0, 0)

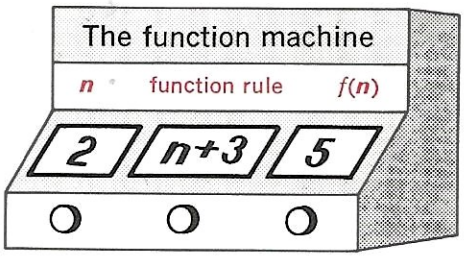


6. Draw two number lines and label them from 0 to 6. Graph each number pair and label it with the letter indicated.

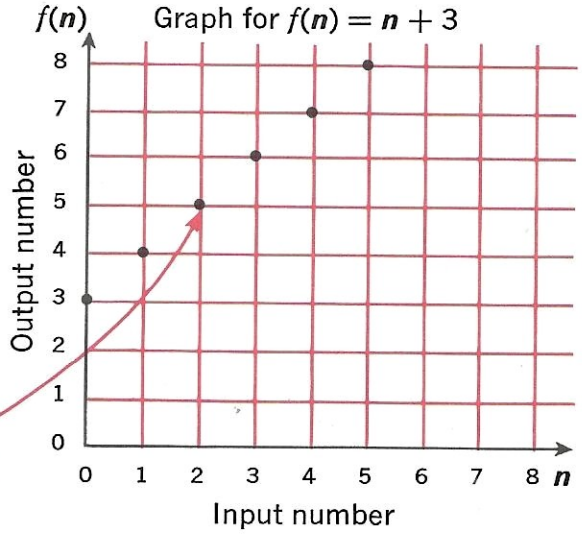
[a] (5, 2), *A* [c] (6, 1), *C* [e] (6, 6), *E* [g] (0, 4), *G* [i] (3, 5), *I* [k] (1, 5), *K*
[b] (2, 5), *B* [d] (4, 3), *D* [f] (0, 0), *F* [h] (6, 0), *H* [j] (6, 3), *J* [l] (3, 3), *L*

2-5 GRAPHING FUNCTIONS

For each function rule and set of input numbers, you can find a set of number pairs. In each number pair, the first number is the input number n and the second number is the output number $f(n)$. You can draw graphs for these sets of number pairs. Think of the numbers on one number line as input numbers and the numbers on the other number line as output numbers. Often, if you graph enough points, you can discover interesting patterns for a given function rule. In the example below notice that the points all lie on one line.



function rule $n + 3$	
n	$f(n)$
0	3
1	4
2	5
3	6
4	7



Exercises

1. Copy and complete the function table. List all the number pairs for the table.

2. Graph the set of number pairs on a graph similar to the one in the example above.

function rule $n - 3$		number pairs
n	$f(n)$	
3	0	→ (3, 0)
4	1	→ (4, 1)
5	2	→ (,)
6		→ (,)
7		→ (,)
8		→ (,)
9		→ (,)

3. Copy and complete the table. Then graph the number pairs.

[a]

function rule $n + 2$	
n	$f(n)$
0	
1	
2	
3	
4	
5	
6	
7	

[b]

function rule $n + 0$	
n	$f(n)$
0	
1	
2	
3	
4	
5	
6	
7	

[c]

function rule $n - 2$	
n	$f(n)$
2	
3	
4	
5	
6	
7	
8	
9	

[d]

function rule $6 - n$	
n	$f(n)$
6	
5	
4	
3	
2	
1	

[e]

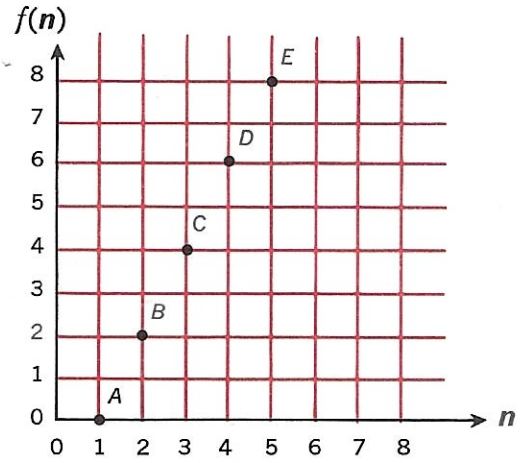
function rule $(2 \cdot n) + 1$	
n	$f(n)$
0	
1	
2	
3	
4	
5	

[f]

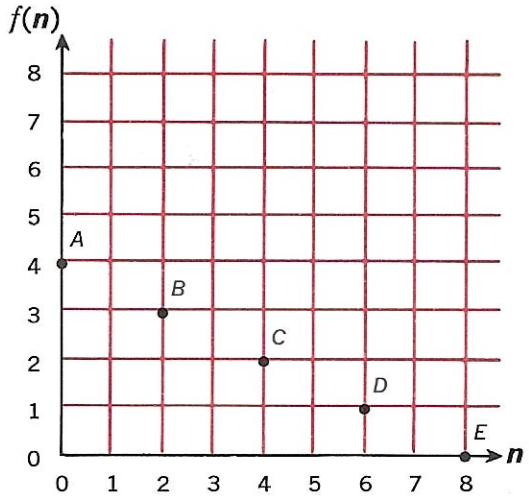
function rule $(24 \div n) - 2$	
n	$f(n)$
2	
3	
4	
6	
8	
12	

★**4.** Use the graph to complete the table.

function rule		
	n	$f(n)$
A		
B		
C		
D		
E		



★**5.** Use the graph to complete the table. Find a function rule as you did in Exercise 4.



function rule

	n	$f(n)$
A		
B		
C		
D		
E		

Think

1. Study the number pattern and try to give the last four sums without adding.

$1 + 3 = 4$

$1 + 3 + 5 = 9$

$1 + 3 + 5 + 7 = 16$

$1 + 3 + 5 + 7 + 9 = 25$

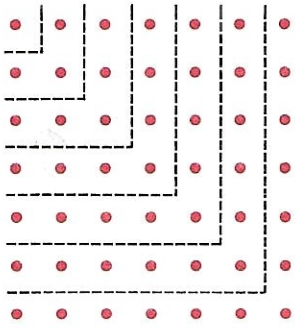
$1 + 3 + 5 + 7 + 9 + 11 = 36$

$1 + 3 + 5 + 7 + 9 + 11 + 13 =$ ▒

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 =$ ▒

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 =$ ▒

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 =$ ▒



2. Explain how the dot figure helps you understand this pattern.

2-6 OPERATIONS AND WORD PROBLEMS

It is important to understand the relationships between operations. It is also important to know when to use certain operations. After you have read a word problem, you should think about what you have read and decide which operation or operations you should use to solve the problem. There are no set rules for doing this. It simply takes careful reading, much thought, and practice. It often helps to visualize sets of objects or to draw pictures. But there are no substitutes for thinking and analyzing the problem. Think about what is given and what you want to find. To help you practice this skill, the following problems are written without numbers. You need only tell what operation or operations to use in order to solve the problem. Think carefully.

Short Stories

In the following short stories, you can think of the numerals as being covered. Give the operation or operations you would use to solve the problem.

1. ▒ boys playing. ▒ boys on the bench. How many boys in all?

2. ▒ girls leaving. ▒ boys leaving. How many students leaving?

3. ▒ marbles in a box. ▒ sacks to fill. Same number in each. How many in each sack?

4. ▒ students. Same number of boys as girls. How many girls?

5. ▒ tennis balls. ▒ cents each. How much for the tennis balls?

6. ▒ boards. ▒ are too short. How many are not too short?

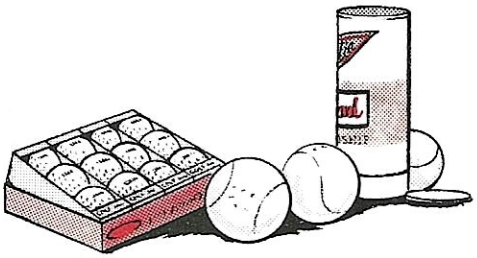
7. ▒ students in class. ▒ students in the library. How many more in the library than in class?

8. ▒ seats in each row and ▒ rows of seats. How many seats?

9. ▒ candy bars given out. ▒ to each person. How many people?

10. ▒ pear trees. ▒ peach trees. ▒ plum trees.
How many trees?

11. ▒ sandwiches at ▒ cents for each. ▒ ice cream bars at ▒ cents each. How much less for the ice cream bars?

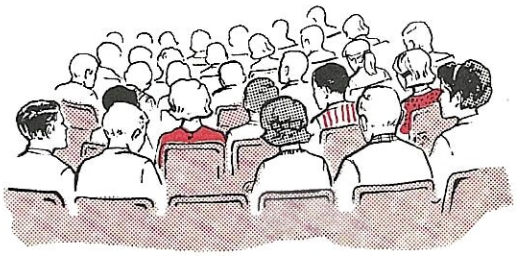


12. ▮▮ cents for ▮▮ golf balls. ▮▮ cents for ▮▮ tennis balls. How much more for a golf ball?

13. Brown's vacation: ▮▮ weeks. Smith's vacation: ▮▮ weeks. How many more days did Smith have?

14. ▮▮ girls and ▮▮ boys in each club. ▮▮ clubs. No student in more than one club. How many students in all the clubs together?

15. ▮▮ rows of seats with ▮▮ seats in each row. ▮▮ students sitting in the seats. How many empty seats?



Exercises

For each problem, give the operation or operations you would use to find the answer. Where more than one operation is needed, give the operations in the order you would use them.

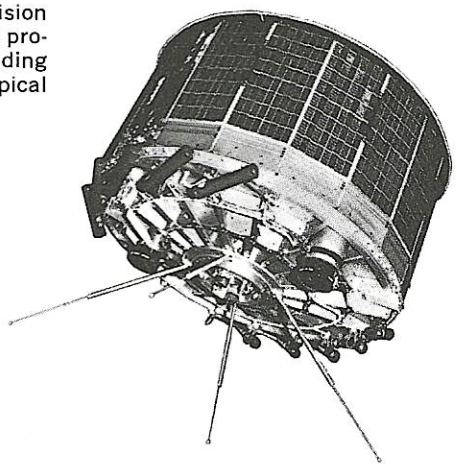
1. If you knew how many marbles each of two boys had, how would you decide how many more one had than the other?
2. If you knew how far a family taking a trip had driven on each of two days, how would you decide how far they had driven in all?
3. If you knew how many hours a friend had worked on Saturday and how much he had earned each hour, how would you compute his Saturday earnings?
4. If you knew how many hours a man worked each day and how much he earned each day, how would you find his hourly wage?
5. If you know the height of a pile of boards and the number of boards in the pile, and if you know that all boards are of the same thickness, how do you determine the thickness of each board?
6. If you know the number of students in each class in a school, how do you find the total school enrollment?
7. If, for a certain class, you know the number of students absent and also the number present, how do you arrive at the total class enrollment?

8. Mary knows the price of a new bicycle. She counts her money and finds that she does not have enough. How can she find out how much more she needs?

9. You have a collection of nickels and dimes. After counting the nickels and the dimes, how do you decide how much money you have in cents?

10. You know the average speed of a satellite in miles per minute and you know how many hours it has been circling the earth. How do you decide how many miles it has traveled?

The 285-pound TIROS (television infrared observation satellite) is programed to photograph the breeding areas of hurricanes and tropical storms.



Think

Here is a counting game. You can name any number from 1 to 6. Your opponent can add to this number any number from 1 to 6. You can add to this total any number from 1 to 6. Whoever reaches 31 first wins the game. There is a way to play so that the first player will always win. Try to find it.

2-7 THE COMMUTATIVE AND ASSOCIATIVE PRINCIPLES

The table shows two important basic principles for multiplication and addition of whole numbers.

Addition	Multiplication
Commutative (order) principle	
We can change the order of the addends, and the sum is the same. Examples: $6 + 8 = 8 + 6$ $175 + 26 = 26 + 175$ In general, if a and b are whole numbers, $a + b = b + a$	We can change the order of the factors, and the product is the same. Examples: $6 \cdot 8 = 8 \cdot 6$ $175 \cdot 26 = 26 \cdot 175$ In general, if a and b are whole numbers, $a \cdot b = b \cdot a$
Associative (grouping) principle	
We can change the grouping of the addends, and the sum is the same. Examples: $(6 + 4) + 5 = 6 + (4 + 5)$ $31 + (7 + 16) = (31 + 7) + 16$ In general, if a, b, and c are whole numbers, $a + (b + c) = (a + b) + c$	We can change the grouping of the factors, and the product is the same. Examples: $(6 \cdot 4) \cdot 5 = 6 \cdot (4 \cdot 5)$ $31 \cdot (7 \cdot 16) = (31 \cdot 7) \cdot 16$ In general, if a, b, and c are whole numbers, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

In actual practice, the most important use of the commutative and associative principles occurs when they are used together. The commutative principle assures us that **we can change the order of the addends and still get the same sum.** The associative principle assures us that **we can change the grouping of the addends and still get the same sum.** The two principles together assure us that **we can arrange three or more addends in any convenient way and still get the same sum.** Similar statements apply for products and factors.

Exercises

1. Solve the equations.

[a] $37 + 26 = n + 37$
[b] $51 \cdot 24 = 24 \cdot a$
[c] $c + 39 = 39 + 57$
[d] $81 + x = 43 + 81$
[e] $(56 + 2) + 13 = 56 + (d + 13)$
[f] $8 \cdot (7 \cdot 6) = (b \cdot 7) \cdot 6$
[g] $(43 + r) + 20 = 43 + (26 + 20)$
[h] $(51 \cdot 3) \cdot 6 = 51 \cdot (3 \cdot t)$
[i] $2641 + a = 57 + 2641$
[j] $6289 \cdot 52 = 52 \cdot m$
[k] $(4 \cdot 3264) \cdot 7 = 4 \cdot (s \cdot 7)$
[l] $6 + (156 + 38) = (n + 156) + 38$

2. Use the commutative principle and the table to find each sum and product.

[a] $1267 + 94$
[b] $131 + 7482$
[c] $978 + 623$
[d] $956 + 486$
[e] $325 + 564$
[f] $39 + 5826$
[g] $8 \cdot 271$
[h] $4 \cdot 941$
[i] $267 \cdot 8$
[j] $35 \cdot 67$
[k] $64 \cdot 27$
[l] $56 \cdot 68$

$623 + 978 = 1601$	$27 \cdot 64 = 1728$
$564 + 325 = 889$	$8 \cdot 267 = 2136$
$94 + 1267 = 1361$	$941 \cdot 4 = 3764$
$5826 + 39 = 5865$	$68 \cdot 56 = 3808$
$486 + 956 = 1442$	$271 \cdot 8 = 2168$
$7482 + 131 = 7613$	$67 \cdot 35 = 2345$

3. Use the commutative and associative principles to find each sum and product.

[a] $99 + (47 + 1)$
[b] $999 + (356 + 1)$
[c] $96 + (79 + 4)$
[d] $990 + (328 + 10)$
[e] $80 + (65 + 20)$
[f] $(200 + 653) + 800$
[g] $(50 + 64) + 50$
[h] $(500 + 647) + 500$
[i] $(75 + 89) + 25$
[j] $(750 + 896) + 250$
[k] $5 \cdot (13 \cdot 2)$
[l] $(25 \cdot 41) \cdot 4$
[m] $25 \cdot (37 \cdot 4)$
[n] $(50 \cdot 97) \cdot 2$
[o] $(125 \cdot 93) \cdot 8$

4. Copy the exercise, filling in each ▒ so that the resulting number sentence is true.

[a] $7 + (9 + 3) = (7 + \blacksquare) + 3$
[b] $8 + (6 + \blacksquare) = (8 + 6) + 4$
[c] $(x + 6) + r = x + (6 + \blacksquare)$
[d] $a + (b + c) = (\blacksquare + \blacksquare) + c$
[e] $\blacksquare + 7 = 8 + (x + 7)$
[f] $(x + \blacksquare) + y = \blacksquare + (r + \blacksquare)$
[g] $7 \cdot (9 \cdot 3) = (7 \cdot \blacksquare) \cdot 3$
[h] $8 \cdot (6 \cdot \blacksquare) = (8 \cdot 6) \cdot 4$
[i] $(n \cdot 6) \cdot r = n \cdot (6 \cdot \blacksquare)$
[j] $a \cdot (b \cdot c) = (\blacksquare \cdot \blacksquare) \cdot c$
[k] $\blacksquare \cdot 7 = 8 \cdot (n \cdot 7)$
[l] $(a \cdot \blacksquare) \cdot b = \blacksquare \cdot (r \cdot \blacksquare)$

5. Tell whether each example illustrates the commutative property, the associative property, or both properties.

[a] $8 + 7 = 7 + 8$
[b] $5 + (6 + 2) = (6 + 2) + 5$
[c] $(4 + 3) + 1 = 4 + (3 + 1)$
[d] $(a + 7) + b = a + (b + 7)$
[e] $(r + s) + t = t + (s + r)$
[f] $8 \cdot 7 = 7 \cdot 8$
[g] $5 \cdot (6 \cdot 2) = (6 \cdot 2) \cdot 5$
[h] $(3 \cdot 4) + 6 = 6 + (3 \cdot 4)$
[i] $(r + s) \cdot t = t \cdot (r + s)$
[j] $(m \cdot n) \cdot p = p \cdot (m \cdot n)$

★**6.** Both addition and multiplication are associative. Do you think subtraction is associative? That is, is it true that if the subtractions can be performed, then for all numbers a, b, and c,

$$a - (b - c) = (a - b) - c?$$

Answer this same question for division. Is it true that for all numbers a, b, and c,

$$a \div (b \div c) = (a \div b) \div c?$$

★**7.** Solve the equations.

[a] $(2 \cdot a) + 7 = 17$
[b] $(3 \cdot m) + 6 = 6$
[c] $18 = (5 \cdot r) + 3$
[d] $30 = (8 \cdot a) - 2$
[e] $b = 16 - b$
[f] $(20 \div s) + 2 = 7$
[g] $19 = (t \cdot 3) + 1$
[h] $2 \cdot s = s + 10$
[i] $c = (5 \cdot c) - 4$
[j] $(3 \cdot x) + (4 \cdot x) = 21$
[k] $36 = r \cdot r$
[l] $28 = n + n + n + n$
[m] $2q + 3 = 3q + 1$
[n] $9x - 9 = 0$
[o] $16 \div p = 1$
[p] $r \div (r + 4) = 0$
[q] $15 = (2 \cdot s) + s$
[r] $15 = (2 \cdot s) - s$

Think

There are many interesting counting games. One can be played with 13 counters laid out in a row.

Two persons play against each other. Each one must, at his turn to play, take one, two, or three counters. The player who is forced to remove the last counter loses the game.

1. If there are five counters left and it is your turn to play, will you win or will you lose?
2. If there are seven counters left and it is your turn, how many should you take in order to win?

2-8 PROPERTIES OF ZERO AND ONE

The number zero plays a very special role in addition. Zero is called the **identity element** for addition of whole numbers. The following examples will help you see why it is given this name.

$$6 + 0 = 6, \quad 100 + 0 = 100, \quad 0 + 15 = 15, \quad 0 + 34 = 34$$

No matter what whole number is added to zero, the sum is that same whole number. In general,

for each whole number a,
$$a + 0 = 0 + a = a$$

The number one plays the same role in multiplication as zero does in addition. The number 1 is called the **identity element** for multiplication of whole numbers. The following examples will help you see why it is given this name.

$$6 \cdot 1 = 6, \quad 100 \cdot 1 = 100, \quad 1 \cdot 15 = 15, \quad 1 \cdot 34 = 34$$

No matter what whole number is multiplied by one, the product is that same whole number. In general,

for each whole number a,
$$a \cdot 1 = 1 \cdot a = a$$

Zero also plays a special role in multiplication and division. It is particularly important to understand why we do not divide by zero. You can best understand this by using the relationship between multiplication and division.

A Suppose the product of two numbers is zero, and one factor is 7. Can you find the other factor?

P F F
$0 = 7 \cdot n$ ⟹ P F F
$0 \div 7 = n$

B Suppose the product of two numbers is zero, and one factor is zero. Can you find the other factor?

P F F
$0 = 0 \cdot n$ ⟹ $n = ?$

Can $n = 7$ or 1 or 2 or 50?

Since n could be **any** number, we agree not to divide zero by zero. We never write "$0 \div 0$."

(C) Now suppose (for a moment) that the product of two numbers is 7, and one of the factors is zero.

P F F

$7 = 0 \cdot n$ ⟹ $n =$??

Something is wrong. There is no such number n.

We can divide zero by every whole number except zero.
We do not divide any number by zero.

Discussion Exercises

1. Think about two addends and the sum and then explain why $a - 0 = a$ and $a - a = 0$.

2. Think about two factors and the product and then explain why $a \div 1 = a$ and $a \div a = 1$.

Exercises

1. Find the sums and differences.

[a] $74 + 0$ [b] $0 + 0$ [c] $96 - 0$ [d] $753 - 753$ [e] $x + 0$ [f] $y - y$ [g] $73 + (86 - 86)$ [h] $(84 - 84) + 7$ [i] $17 + (x - x)$

2. Solve the equations.

[a] $67 + x = 67$ [b] $92 - x = 0$ [c] $74 - x = 74$ [d] $x - 43 = 0$ [e] $x - 0 = 99$ [f] $87 + (x - 12) = 87$ [g] $96 - (x + 96) = 0$ [h] $x + x = 0$ [i] $0 + x = 0$

3. Find the products and quotients.

[a] $74 \cdot 1$ [b] $1 \cdot 1$ [c] $96 \div 1$ [d] $753 \div 753$ [e] $n \cdot 0$ [f] $n \div n$ [g] $73 \cdot (86 \div 86)$ [h] $(84 \div 84) \cdot 7$ [i] $17 \cdot (n \div n)$

4. Solve the equations.

[a] $67 \cdot a = 67$ [b] $92 \div a = 1$ [c] $74 \div a = 74$ [d] $a \div 43 = 1$ [e] $a \div 1 = 99$ [f] $87 \cdot (a \div 12) = 87$ [g] $96 \div (a \cdot 96) = 1$ [h] $a \cdot a = 1$ [i] $a \cdot (a + 1) = a$

5. Find the products and quotients.

[a] $793 \cdot 0$ [c] $75 \cdot (a - a)$ [e] $(7 + 7) \cdot (7 - 7)$

[b] $0 \div 432$ [d] $(x - x) \div 17$ [f] $(16 \cdot 0) \div 8$

6. Copy each problem. Write P (product) or F (factor) above each number. (Think about the expressions in the gray screen as a single number.) If it is possible to do so, determine n.

[a] $83 \cdot n = 0$ [d] $(7 - n) \cdot n = 0$ [g] $n \cdot 0 = 0$

[b] $n \div 14 = 0$ [e] $(7 + n) \cdot n = 0$ [h] $n \cdot 75 = n$

[c] $(7 - n) \cdot 12 = 0$ [f] $n \div a = 0$ [i] $n \div 75 = n$

7. Which of the expressions do not represent numbers?

[a] $0 \div 6$ [c] $6 \div 6$

[b] $6 \div 0$ [d] $0 \div 0$

8. Suppose two students both choose a whole number. What can you say about the numbers if their product is zero? What can you say if their product is 1? If their product is 2? 3? 4? 5?

Think

In numbering the pages of a book, 495 digits were used. How many pages were there? How many 9's were used?

2-9 THE DISTRIBUTIVE PRINCIPLE

The distributive principle forms the link between addition and multiplication. One easy way to understand the distributive principle is to think of writing a factor as a sum before multiplying.

Distributive principle

Multiplication distributes over addition.

Examples: $6 \cdot 5 = (4 \cdot 5) + (2 \cdot 5)$

$3 \cdot 12 = (3 \cdot 10) + (3 \cdot 2)$

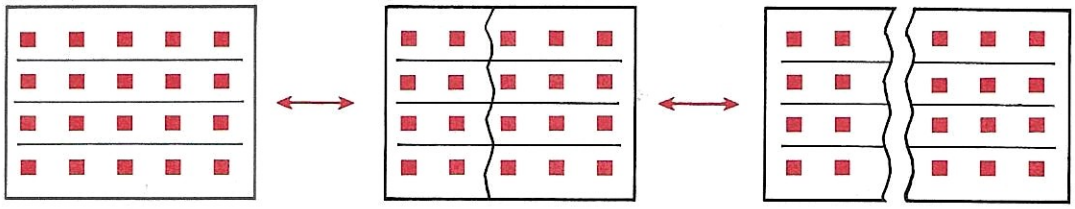
In general, if a, b, and c are whole numbers,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

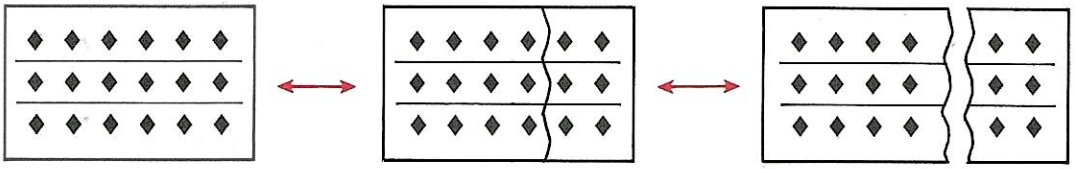
The figures in Exercise 1 will help you understand the distributive principle.

Exercises

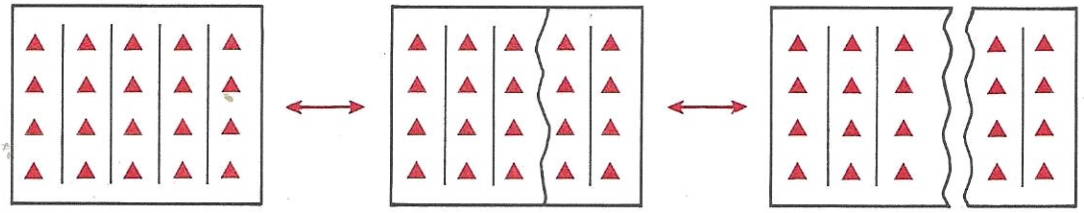
1. Study each picture. Then solve the equation.

[a] 

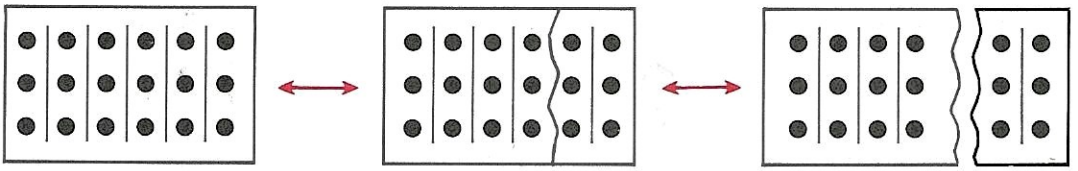
$4 \cdot 5 = 4 \cdot (2 + 3) = (n \cdot 2) + (4 \cdot 3)$

[b] 

$3 \cdot 6 = 3 \cdot (4 + 2) = (3 \cdot n) + (3 \cdot 2)$

[c] 

$5 \cdot 4 = (3 + 2) \cdot 4 = (3 \cdot 4) + (n \cdot 4)$

[d] 

$6 \cdot 3 = (4 + 2) \cdot 3 = (4 \cdot n) + (2 \cdot 3)$

2. Solve the equations.

[a] $5 \cdot 7 = n \longrightarrow 5 \cdot (4 + 3) = n \longrightarrow (5 \cdot 4) + (5 \cdot 3) = n$
[b] $4 \cdot 5 = a \longrightarrow 4 \cdot (1 + 4) = a \longrightarrow (4 \cdot 1) + (4 \cdot 4) = a$
[c] $6 \cdot 8 = c \longrightarrow 6 \cdot (4 + 4) = c \longrightarrow (6 \cdot 4) + (6 \cdot 4) = c$
[d] $3 \cdot 6 = x \longrightarrow 3 \cdot (5 + 1) = x \longrightarrow (3 \cdot 5) + (3 \cdot 1) = x$
[e] $8 \cdot 9 = b \longrightarrow 8 \cdot (2 + 7) = b \longrightarrow (8 \cdot 2) + (8 \cdot 7) = b$
[f] $7 \cdot 8 = m \longrightarrow 7 \cdot (6 + 2) = m \longrightarrow (7 \cdot 6) + (7 \cdot 2) = m$

3. Solve the equations.

[a] $5 \cdot 8 = 5 \cdot (6 + a) \longrightarrow 5 \cdot 8 = (5 \cdot 6) + (5 \cdot a)$
[b] $6 \cdot 4 = 6 \cdot (n + 1) \longrightarrow 6 \cdot 4 = (6 \cdot n) + (6 \cdot 1)$
[c] $4 \cdot 9 = 4 \cdot (3 + b) \longrightarrow 4 \cdot 9 = (4 \cdot 3) + (4 \cdot b)$
[d] $7 \cdot 6 = 7 \cdot (y + 2) \longrightarrow 7 \cdot 6 = (7 \cdot y) + (7 \cdot 2)$
[e] $3 \cdot 8 = 3 \cdot (4 + r) \longrightarrow 3 \cdot 8 = (3 \cdot 4) + (3 \cdot r)$
[f] $8 \cdot 7 = 8 \cdot (d + 1) \longrightarrow 8 \cdot 7 = (8 \cdot d) + (8 \cdot 1)$
[g] $2 \cdot 5 = 2 \cdot (3 + t) \longrightarrow 2 \cdot 5 = (2 \cdot 3) + (2 \cdot t)$
[h] $9 \cdot 8 = 9 \cdot (m + 3) \longrightarrow 9 \cdot 8 = (9 \cdot m) + (9 \cdot 3)$

4. Solve the equations.

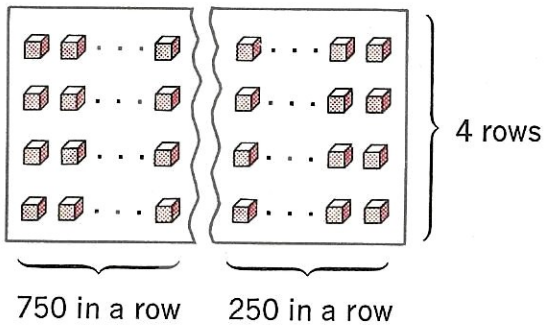
[a] $(6 \cdot 5) + (6 \cdot 1) = 6 \cdot a$
[b] $(5 \cdot 3) + (5 \cdot 4) = 5 \cdot m$
[c] $(9 \cdot 2) + (9 \cdot 8) = 9 \cdot d$
[d] $(6 \cdot 4) + (6 \cdot 5) = 6 \cdot q$
[e] $(8 \cdot 10) + (8 \cdot 3) = 8 \cdot n$
[f] $(5 \cdot 10) + (5 \cdot 7) = 5 \cdot s$
[g] $(6 \cdot 30) + (6 \cdot 5) = 6 \cdot y$
[h] $(4 \cdot 70) + (4 \cdot 8) = 4 \cdot r$
[i] $(7 \cdot 200) + (7 \cdot 50) + (7 \cdot 3) = 7 \cdot a$
[j] $(6 \cdot 600) + (6 \cdot 10) + (6 \cdot 9) = 6 \cdot c$
[k] $(8 \cdot 400) + (8 \cdot 30) + (8 \cdot 8) = 8 \cdot m$
[l] $(5 \cdot 900) + (5 \cdot 90) + (5 \cdot 9) = 5 \cdot q$
[m] $(6 \cdot 4000) + (6 \cdot 700) + (6 \cdot 50) + (6 \cdot 3) = 6 \cdot b$
[n] $(3 \cdot 6000) + (3 \cdot 100) + (3 \cdot 20) + (3 \cdot 8) = 3 \cdot e$
[o] $(8 \cdot 3000) + (8 \cdot 800) + (8 \cdot 90) + (8 \cdot 6) = 8 \cdot q$

5. The distributive principle is sometimes described by saying, "Multiplication is distributive over addition." Now check the following subtraction statements to see if they are correct.

[a] $5 \cdot (8 - 3) = (5 \cdot 8) - (5 \cdot 3)$
[b] $6 \cdot (7 - 2) = (6 \cdot 7) - (6 \cdot 2)$
[c] $9 \cdot (5 - 4) = (9 \cdot 5) - (9 \cdot 4)$
[d] $4 \cdot (7 - 6) = (4 \cdot 7) - (4 \cdot 6)$

Do you think multiplication is distributive over subtraction?

6. The small black dots are to indicate that many cubes have been left out. Give the total number of cubes, without pencil-and-paper computation.

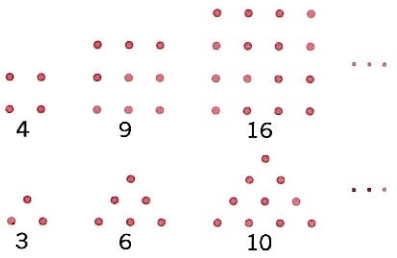


7. Copy and complete the equations.

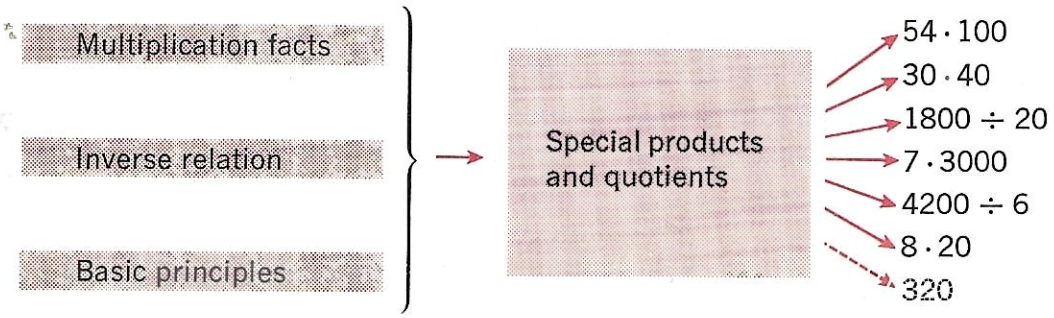
[a] $a \cdot (b + c) = (a \cdot b) + \square$
[b] $s \cdot (r + t) = \square + (s \cdot t)$
[c] $\square \cdot (c + d) = (n \cdot c) + (n \cdot d)$
[d] $b \cdot (\square + n) = (b \cdot m) + (b \cdot n)$
[e] $r \cdot (a + \square) = (r \cdot a) + (r \cdot y)$
[f] $f \cdot (g + h) = \square + \square$
[g] $\square = (a \cdot n) + (a \cdot m)$
[h] $\square = (b \cdot s) + (b \cdot s)$

Think

The dots show why some numbers are called square numbers and some are called triangular. Find one number that is both square and triangular. Show the square and triangular arrays of dots for this number.



2-10 SPECIAL PRODUCTS AND QUOTIENTS



By using the basic multiplication facts, the inverse relation between multiplication and division, and the basic principles, we can find many special products and quotients. The following exercises illustrate this point.

Exercises

1. Find the products.

[a] 5 · 10 [b] 10 · 7 [c] 8 · 100 [d] 2 · 10 [e] 100 · 6 [f] 3 · 100 [g] 9 · 10 [h] 10 · 4 [i] 100 · 7 [j] 5 · 100 [k] 9 · 1000 [l] 1000 · 40

2. Find the number for ***a***. Then find the number for ***b***.

[a] $10 \cdot 10 = a \longrightarrow 7 \cdot 10 \cdot 10 = b$
[b] $7 \cdot 10 \cdot 10 = a \longrightarrow 70 \cdot 10 = b$
[c] $10 \cdot 100 = a \longrightarrow 4 \cdot 10 \cdot 100 = b$
[d] $4 \cdot 10 \cdot 100 = a \longrightarrow 40 \cdot 100 = b$
[e] $6 \cdot 10 \cdot 10 = a \longrightarrow 60 \cdot 10 = b$
[f] $9 \cdot 10 \cdot 100 = a \longrightarrow 90 \cdot 100 = b$
[g] $3 \cdot 10 \cdot 10 = a \longrightarrow 30 \cdot 10 = b$
[h] $8 \cdot 10 \cdot 100 = a \longrightarrow 80 \cdot 100 = b$
[i] $2 \cdot 100 \cdot 100 = a \longrightarrow 200 \cdot 100 = b$
[j] $9 \cdot 100 \cdot 100 = a \longrightarrow 900 \cdot 100 = b$

3. Find the products.

[a] $40 \cdot 10$
[b] $10 \cdot 30$
[c] $60 \cdot 100$
[d] $100 \cdot 70$
[e] $80 \cdot 10$
[f] $80 \cdot 100$
[g] $50 \cdot 100$
[h] $50 \cdot 1000$
[i] $1000 \cdot 60$
[j] $70 \cdot 1000$

4. Find the number for ***a***. Then find the number for ***b***.

[a] $(10 \cdot 20) + (10 \cdot 3) = a \longrightarrow 10 \cdot 23 = b$
[b] $(10 \cdot 60) + (10 \cdot 7) = a \longrightarrow 10 \cdot 67 = b$
[c] $(10 \cdot 40) + (10 \cdot 3) = a \longrightarrow 10 \cdot 43 = b$
[d] $(10 \cdot 70) + (10 \cdot 6) = a \longrightarrow 10 \cdot 76 = b$
[e] $(10 \cdot 80) + (10 \cdot 2) = a \longrightarrow 10 \cdot 82 = b$
[f] $(100 \cdot 20) + (100 \cdot 8) = a \longrightarrow 100 \cdot 28 = b$
[g] $(100 \cdot 40) + (100 \cdot 7) = a \longrightarrow 100 \cdot 47 = b$
[h] $(100 \cdot 50) + (100 \cdot 3) = a \longrightarrow 100 \cdot 53 = b$
[i] $(100 \cdot 90) + (100 \cdot 1) = a \longrightarrow 100 \cdot 91 = b$
[j] $(100 \cdot 10) + (100 \cdot 6) = a \longrightarrow 100 \cdot 16 = b$

5. Find the products.

[a] $53 \cdot 10$
[b] $10 \cdot 28$
[c] $36 \cdot 10$
[d] $10 \cdot 95$
[e] $100 \cdot 48$
[f] $56 \cdot 100$
[g] $10 \cdot 72$
[h] $83 \cdot 100$
[i] $100 \cdot 65$
[j] $72 \cdot 100$

6. Find the number for ***a***. Then find the number for ***b***.

[a] $4 \cdot 3 \cdot 10 = a \longrightarrow 4 \cdot 30 = b$
[b] $5 \cdot 7 \cdot 10 = a \longrightarrow 5 \cdot 70 = b$
[c] $3 \cdot 8 \cdot 10 = a \longrightarrow 3 \cdot 80 = b$
[d] $6 \cdot 4 \cdot 10 = a \longrightarrow 6 \cdot 40 = b$
[e] $8 \cdot 6 \cdot 10 = a \longrightarrow 8 \cdot 60 = b$
[f] $7 \cdot 3 \cdot 100 = a \longrightarrow 7 \cdot 300 = b$
[g] $8 \cdot 6 \cdot 100 = a \longrightarrow 8 \cdot 600 = b$
[h] $5 \cdot 9 \cdot 100 = a \longrightarrow 5 \cdot 900 = b$
[i] $8 \cdot 8 \cdot 100 = a \longrightarrow 8 \cdot 800 = b$
[j] $9 \cdot 2 \cdot 100 = a \longrightarrow 9 \cdot 200 = b$

7. Find the products.

[a] $7 \cdot 20$
[b] $6 \cdot 70$
[c] $30 \cdot 2$
[d] $80 \cdot 3$
[e] $5 \cdot 70$
[f] $8 \cdot 200$
[g] $9 \cdot 400$
[h] $70 \cdot 7$
[i] $300 \cdot 4$
[j] $6 \cdot 900$

8. Find the numbers for a, b, and c.

[a] $(3 \cdot 4) \cdot (10 \cdot 10) = a \longrightarrow (3 \cdot 10) \cdot (4 \cdot 10) = b \longrightarrow 30 \cdot 40 = c$

[b] $(7 \cdot 2) \cdot (10 \cdot 10) = a \longrightarrow (7 \cdot 10) \cdot (2 \cdot 10) = b \longrightarrow 70 \cdot 20 = c$

[c] $(5 \cdot 3) \cdot (10 \cdot 10) = a \longrightarrow (5 \cdot 10) \cdot (3 \cdot 10) = b \longrightarrow 50 \cdot 30 = c$

[d] $(4 \cdot 6) \cdot (10 \cdot 100) = a \longrightarrow (4 \cdot 10) \cdot (6 \cdot 100) = b \longrightarrow 40 \cdot 600 = c$

[e] $(2 \cdot 9) \cdot (10 \cdot 100) = a \longrightarrow (2 \cdot 10) \cdot (9 \cdot 100) = b \longrightarrow 20 \cdot 900 = c$

[f] $(6 \cdot 8) \cdot (10 \cdot 10) = a \longrightarrow (6 \cdot 10) \cdot (8 \cdot 10) = b \longrightarrow 60 \cdot 80 = c$

[g] $(9 \cdot 4) \cdot (10 \cdot 100) = a \longrightarrow (9 \cdot 10) \cdot (4 \cdot 100) = b \longrightarrow 90 \cdot 400 = c$

[h] $(7 \cdot 8) \cdot (100 \cdot 100) = a \longrightarrow (7 \cdot 100) \cdot (8 \cdot 100) = b \longrightarrow 700 \cdot 800 = c$

[i] $(6 \cdot 4) \cdot (10 \cdot 100) = a \longrightarrow (6 \cdot 10) \cdot (4 \cdot 100) = b \longrightarrow 60 \cdot 400 = c$

9. Find the quotients.

[a] Since $60 \cdot 30 = 1800$, we know that $1800 \div 30 = a$.

[b] Since $40 \cdot 20 = 800$, we know that $800 \div 20 = c$.

[c] Since $70 \cdot 40 = 2800$, we know that $2800 \div 40 = x$.

[d] Since $80 \cdot 20 = 1600$, we know that $1600 \div 20 = m$.

[e] Since $50 \cdot 300 = 15{,}000$, we know that $15{,}000 \div 300 = r$.

[f] Since $200 \cdot 70 = 14{,}000$, we know that $14{,}000 \div 70 = d$.

[g] Since $20 \cdot 700 = 14{,}000$, we know that $14{,}000 \div 700 = q$.

[h] Since $400 \cdot 80 = 32{,}000$, we know that $32{,}000 \div 80 = s$.

10. Think about the missing factor. Then give the quotient.

[a] $? \cdot 30 = 600$
$600 \div 30 = n$

[b] $? \cdot 40 = 2400$
$2400 \div 40 = a$

[c] $? \cdot 50 = 1500$
$1500 \div 50 = t$

[d] $r \cdot 60 = 1800$
$1800 \div 60 = r$

[e] $m \cdot 8 = 1600$
$1600 \div 8 = m$

[f] $s \cdot 300 = 1200$
$1200 \div 300 = s$

[g] $q \cdot 90 = 2700$
$2700 \div 90 = q$

[h] $d \cdot 80 = 6400$
$6400 \div 80 = d$

[i] $t \cdot 300 = 21{,}000$
$21{,}000 \div 300 = t$

[j] $a \cdot 80 = 3200$
$3200 \div 80 = a$

[k] $s \cdot 50 = 3500$
$3500 \div 50 = s$

[l] $b \cdot 70 = 2800$
$2800 \div 70 = b$

[m] $r \cdot 40 = 24{,}000$
$24{,}000 \div 40 = r$

[n] $m \cdot 200 = 18{,}000$
$18{,}000 \div 200 = m$

[o] $q \cdot 800 = 40{,}000$
$40{,}000 \div 800 = q$

11. Find the products and quotients.

[a] $3 \cdot 100$
[b] $10 \cdot 67$
[c] $1200 \div 20$
[d] $800 \div 2$
[e] $100 \cdot 80$
[f] $50 \cdot 70$
[g] $1800 \div 100$
[h] $30 \cdot 4$
[i] $6 \cdot 200$
[j] $2400 \div 80$
[k] $90 \cdot 40$
[l] $1600 \div 8$
[m] $4000 \div 50$
[n] $70 \cdot 80$
[o] $600 \cdot 10$
[p] $5600 \div 8$
[q] $12{,}000 \div 40$
[r] $18{,}000 \div 300$
[s] $600 \cdot 70$
[t] $100 \cdot 72$
[u] $6800 \div 68$
[v] $3000 \div 60$
[w] $500 \cdot 40$
[x] $7200 \div 9$

★**12.** Find the products and quotients. Write your answers in scientific notation.

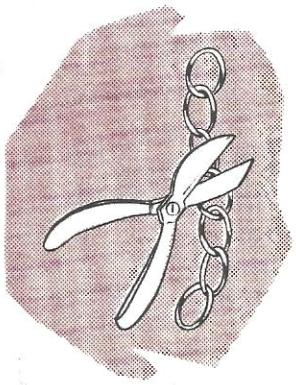
[a] $3 \cdot 10^3 \cdot 2 \cdot 10^4$
[b] $5 \cdot 10^4 \cdot 4 \cdot 10^5$
[c] $4 \cdot 10^6 \div 10^3$
[d] $3 \cdot 10^8 \div 10^8$
[e] $7 \cdot 10^7 \div 7$
[f] $50 \cdot 10^3 \cdot 10^2$
[g] $60{,}000 \div 2 \cdot 10^2$
[h] $100{,}000 \div 5 \cdot 10^4$
[i] $500 \cdot 10^2 \cdot 20 \cdot 10^3$
[j] $3{,}000{,}000 \div 30 \cdot 10^3$
[k] $2 \cdot 10^{10} \div 10 \cdot 10^2$
[l] $10 \cdot 10^4 \cdot 4 \cdot 10^{10}$

★**13.** Write the numbers in scientific notation. Find the product or quotient.

[a] $10{,}000{,}000 \cdot 300{,}000$
[b] $600{,}000{,}000 \div 30{,}000$
[c] $4{,}000{,}000{,}000 \div 200.000$
[d] $400{,}000{,}000 \cdot 2{,}000{,}000$
[e] $5{,}000{,}000 \cdot 80{,}000{,}000$
[f] $30{,}000{,}000{,}000{,}000 \div 30{,}000$

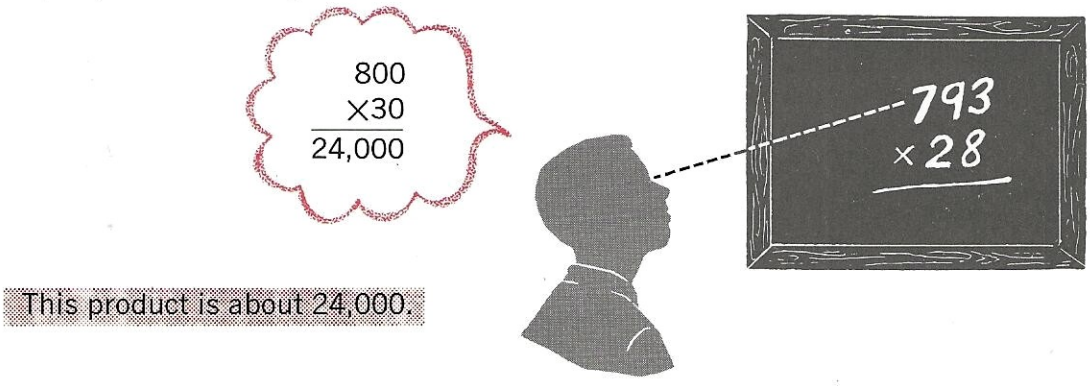
Think

A traveler wishes to stay at an inn for several days. He has no money, but he has a valuable gold chain. The innkeeper agrees to let him stay on the condition that each day he pay one link of his chain. The traveler must pay every day; that is, at the end of one day, the innkeeper must have one link, at the end of two days, two links, and so forth. The traveler must cut his chain in order to make these payments. If the chain has seven links and the traveler stays for seven days, show how by cutting only one link in all he can pay for his lodging every day.



2-11 ESTIMATING

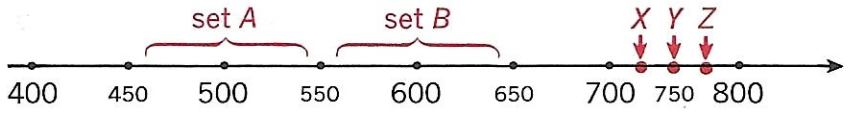
The ability to make careful estimates in mathematics is one of the most useful skills you can develop. Careful estimates will often help you find errors in computation. If someone gave an answer of 7930 for the product shown here, you would know at once that a mistake had been made. The following exercises will help you sharpen your estimating skills.



This product is about 24,000.

Exercises

1. In estimating, we often choose the closest multiple of 10, 100, or 1000 to represent the given numbers.



[a] What is the closest multiple of 100 for the numbers indicated by set *A*?
[b] What is the closest multiple of 100 for the numbers indicated by set *B*?
[c] What is the closest multiple of 100 for the number indicated by *X*?
[d] What is the closest multiple of 100 for the number indicated by *Z*?
[e] For numbers ending in 5, 50, or 500, we usually call the next **higher** multiple of 10, 100, or 1000 the closest multiple. According to this agreement, which multiple of 100 should be used for *Y*?

2. Tell which of the two numbers is the closest multiple of 10, 100, or 1000 for the number given in color.

[a] 70, 78, 80
[b] 30, 31, 40
[c] 50, 56, 60
[d] 100, 193, 200
[e] 600, 629, 700
[f] 500, 514, 600
[g] 100, 149, 200
[h] 800, 865, 900
[i] 200, 213, 300
[j] 6000, 6281, 7000
[k] 5000, 5706, 6000
[l] 2000, 2567, 3000

3. Use multiples of 10, 100, or 1000 for each ▒.

[a] To estimate 387 + 519, we can find the sum 400 + ▒.

[b] To estimate 784 · 8, we can find the product ▒ · 8.

[c] To estimate 63 · 78, we can find the product 63 · ▒.

[d] To estimate 2139 ÷ 69, we can find the quotient 2100 ÷ ▒.

[e] To estimate 692 − 176, we can find the difference ▒ − 200.

[f] To estimate 6 · 4195, we can find the product 6 · ▒.

[g] To estimate 524 + 395, we can find the sum ▒ + 400.

[h] To estimate 6128 − 2716, we can find the difference 6000 − ▒.

[i] To estimate 63 · 295, we can find the product 60 · ▒.

[j] To estimate 78 + 63 + 91, we can find the sum 80 + 60 + ▒.

[k] To estimate 8199 + 2411, we can find the sum ▒ + 2000.

[l] To estimate 381 · 714, we can find the product ▒ · 700.

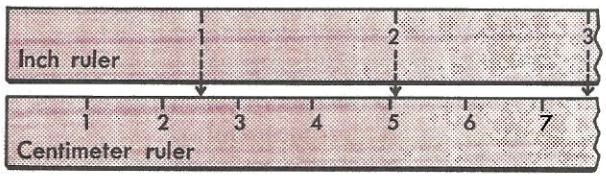
4. Give an estimate for each part of Exercise 3.

5. Give an estimate for each of the following exercises.

[a] 721 + 695	[d] 32 · 77	[g] 1984 + 6911	[j] 625 + 379
[b] 843 + 275	[e] 568 · 4	[h] 78 + 24 + 98	[k] 7684 + 3211
[c] 629 − 179	[f] 71 · 688	[i] 213 · 498	[l] 566 · 32

6. Give an estimate for each problem.

[a] A meter (m) is about 39 in. If a board is 6 m long, about how many inches long is it?



[b] An airplane travels for 4 hours (hr) at an average speed of 587 mph. About how far does it travel?

[c] Sound travels about 1088 ft per sec in air. About how far does it travel in a minute?

[d] There are 5280 ft in a mile. About how many feet are there in 48 mi?

[e] There are about 91 centimeters (cm) in a yard. About how many centimeters long is a 12-ft board?

[f] There are 27 cubic feet (cu ft) in a cubic yard (cu yd). About how many cubic feet are there in 63 cu yd ?

★**7.** Estimate how many seconds old you are?

★**8.** The circumference of the earth at the equator is 24,902 mi. Estimate the circumference in feet.

Think

A man needs to take a fox, a goose, and a sack of corn across a river. He has a boat in which he cannot take more than one across at a time. He cannot leave the fox and the goose together or the goose and the corn, for one will eat the other. How can he get all three across the river, taking one at a time with him in the boat? Draw a diagram to show your solution.

CHAPTER REVIEW

Key Concepts

1. Addition and subtraction are related operations. Adding a number and then subtracting that same number are **inverse** operations.
2. Multiplication and division are related operations. Multiplying by a number and then dividing by that same number are inverse operations.
3. The **function** idea and the operations can be used to obtain sets of number pairs.
4. Sets of numbers can be graphed on the number line. Sets of number pairs can be graphed using a pair of number lines.
5. Addition and multiplication of whole numbers are **commutative** and **associative**. For all whole numbers a, b, and c,

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$
$$a \cdot b = b \cdot a, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

6. Zero is the **identity element** for addition of whole numbers. The number one is the identity element for multiplication of whole numbers. For all whole numbers a,

$$a + 0 = a, \quad a \cdot 1 = a$$

7. We do not divide by zero.
8. Multiplication is **distributive** with respect to addition. For all whole numbers a, b, and c,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

9. The ability to estimate the answer to certain problems is a useful skill. Multiples of 10, 100, 1000 are often used for estimating.

Vocabulary List

add	equation	multiple	product
addend	estimate	multiply	quotient
commutative	graph	number line	solve
difference	identity	operation	subtract
distributive	inverse	parentheses	sum
divide			

Review Exercises

1. In the statement following, give the other subtraction equation: Since $6 + 7 = 13$, we know that $13 - 7 = 6$ and $13 - \blacksquare = \blacksquare$.

2. The same two digits in each equation have been covered. Solve the division equation without finding the digits.
Since $2\blacksquare \cdot 17 = 4692$, we know that $4692 \div 2\blacksquare = n$.

3. Give the missing numbers for each gray screen.

	function rule $n \cdot 7$	
	n	$f(n)$
[a]	6	
[b]	4	
[c]	8	

	function rule $(6 \cdot n) + 3$	
	n	$f(n)$
[d]	3	
[e]	9	
[f]	0	

4. Find a function rule for each table.

[a] function rule ▒▒▒▒

n	$f(n)$
6	9
0	3
8	11
10	13

[b] function rule ▒▒▒▒

n	$f(n)$
2	9
7	19
0	5
5	15

5. Draw a number line and label it 0 through 10. Graph the whole numbers greater than 0 and less than 5.

6. Copy and complete the table. Then graph the set of number pairs.

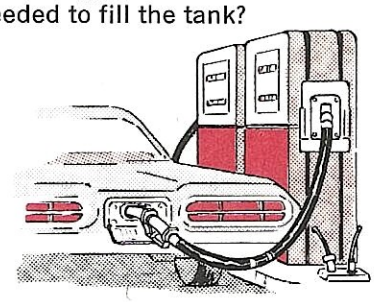
	function rule $5 - n$	
	n	$f(n)$
[a]	5	
[b]	4	
[c]	3	
[d]	2	
[e]	1	
[f]	0	

7. Draw a graph of the following set of number pairs.
$\{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

8. Give the operation for each word problem.

[a] If Mr. Jones drove for ▒▒ hours at a speed of ▒▒ miles per hour, how far did he drive?

[b] A tank that holds ▒▒ gallons of gasoline has only ▒▒ gallons in it. How much is needed to fill the tank?



[c] It took ▒▒ days to complete the job. How many weeks did it take?

[d] There were ▒▒ people remaining in the room after ▒▒ girls left. How many were there before the girls left?

9. For each equation, give the number of the basic principle.

[a] $a + 0 = a$
[b] $(6 + 7) + 9 = 6 + (7 + 9)$
[c] $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
[d] $a + 7 = 7 + a$
[e] $7 \cdot (n + 6) = (7 \cdot n) + (7 \cdot 6)$
[f] $586 \cdot 78 = 78 \cdot 586$
[g] $5280 \cdot 1 = 5280$

1. Commutative principle (+)
2. Commutative principle (×)
3. Associative principle (+)
4. Associative principle (×)
5. Identity for addition
6. Identity for multiplication
7. Distributive principle

10. Find the products and quotients.

[a] $100 \cdot 38$	[g] $100 \cdot 70$
[b] $5270 \div 527$	[h] $600 \div 10$
[c] $3 \cdot 80$	[i] $200 \cdot 70$
[d] $420 \div 6$	[j] $80 \cdot 30$
[e] $5 \cdot 20$	[k] $1800 \div 60$
[f] $1200 \div 6$	[l] $3600 \div 40$

11. Choose the best estimate for each exercise.

[a]	[b]	[c]
516	4182	618
897	-1965	$\times 37$
$+89$		
1400	1000	18,000
1500	2000	21,000
1600	3000	24,000

CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 46–47, 81–87.

———. *Magic House of Numbers.* New York, New American Library of World Literature, 1957; pp. 43–52.

BELL, E. T. *Men of Mathematics.* New York, Simon and Schuster, 1937; pp. 3–34.

BRUMFIEL, C. F., EICHOLZ, R. E., and SHANKS, M.E. *Introduction to Mathematics.* Reading, Massachusetts, Addison-Wesley Publishing Company, 1963; pp. 84–91.

DANTZIG, TOBIAS. *Number—The Language of Science.* Garden City, New York, Doubleday and Company, 1956; pp. 58-77.

HOGBEN, LANCELOT. *The Wonderful World of Mathematics.* Garden City, New York, Doubleday and Company, 1955; pp. 20-25, 44-51.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Invitation to Mathematics.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1960; pp. 7–28.

———. *Sets, Sentences, and Operations. Ibid.*, pp. 1–34.

SMITH, D. E. *History of Mathematics.* New York, Dover Publications, 1953; Vol. II, pp. 32–36, 395–406.

Activities

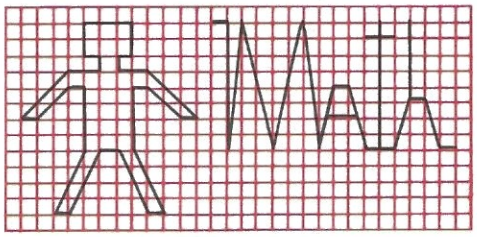
A. Find discussions of special "modular" systems. Write about one of these, setting up rules for the basic operations of addition, subtraction, multiplication, and division.

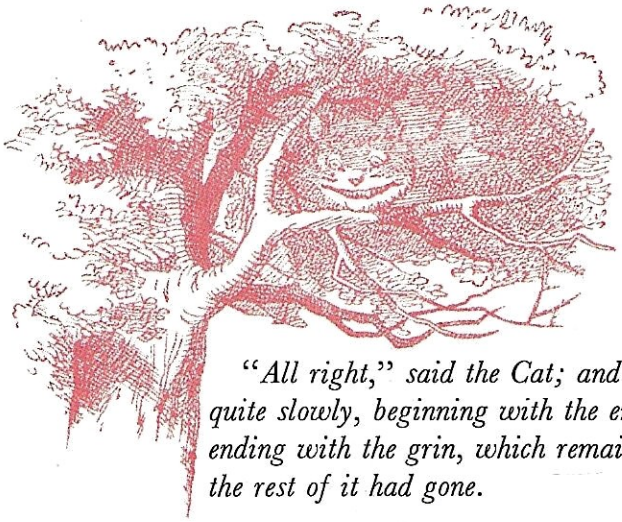
B. Read about one of the great mathematicians and write about a major contribution he has made. (*Invitation to Mathematics* has historical notes on such men as Archimedes, Leibniz, Gauss, Einstein, Descartes, Newton, and von Neumann. See also *Men of Mathematics.*)

C. Graphs are used in many ways (for example, in architecture). Find three purposes for graphs and tell how graphs are used in each situation.

D. Get a map and show how locating a city on a map is similar to finding a point on a graph.

E. Give a set of number pairs such that when the points are graphed and connected in the order given, you will have a picture or a word. For example:





"All right," said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.

3

Geometry Unit I—Points, Lines, Planes

A. NONMETRIC GEOMETRY

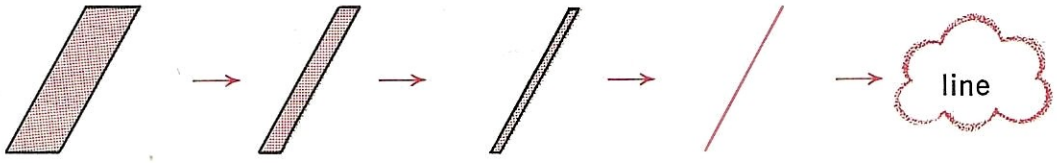
Many parts of geometry are not concerned with measurement. These parts deal with the relations between points, lines, and planes that **do not use numbers**. These geometric ideas are a part of **nonmetric geometry**.

3-1 POINTS AND LINES

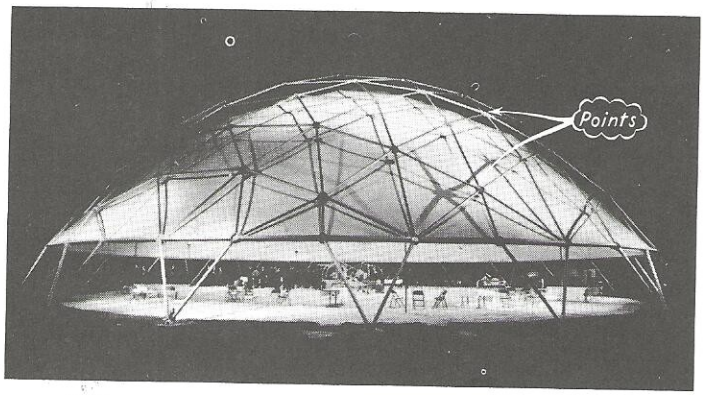
Who has seen a point?

Dot — Small dot — Smaller dot — Smaller dot — An idea

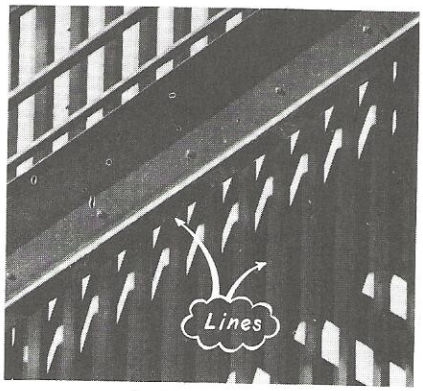
Who has seen a straight line?



Strip — Thin strip — Thinner strip — Thinner strip — An idea



Geodesic dome in Bamako, Mali



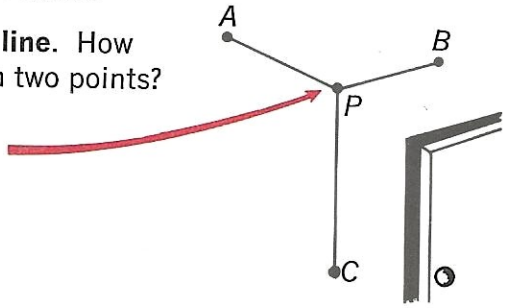
A stairway

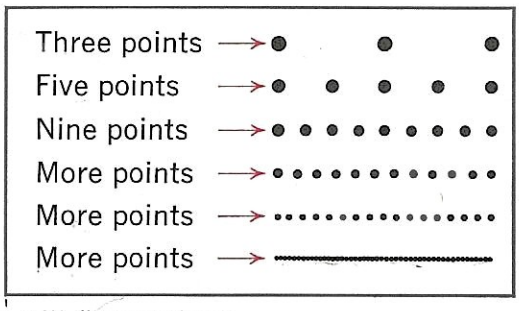
We see things that suggest the ideas of points and straight lines. The pictures we draw suggest these ideas, and we often say "draw a line" when we mean **draw a picture to represent a line**. Or we may say "mark a point" when we mean **draw a dot to represent a point**.

Idea	We draw	We write	We say
point	B• A• •C	A, B, C	point *A* point *B* point *C*
line	m n	*m* *n*	line *m* line *n*
	A B C D	$\overleftrightarrow{AB}$ $\overleftrightarrow{CD}$	line *AB* line *CD*

Discussion Exercises

1. [a] Is the tip of your pencil a point? [b] Is the tip of a pin a point?
2. [a] Is a ray of light a line? [b] Is a thin, tightly stretched wire a line?
3. The objects in Exercises 1 and 2 suggest points, lines, and pieces of lines. Look around your classroom and find other objects that suggest these geometric ideas.
4. When we say "line," we always mean **straight line**. How many different straight lines are there through two points?
5. Look at a point *P* suggested by one corner of your classroom. Think about three other points *A*, *B*, and *C* as indicated in the figure. Points *A* and *P* suggest $\overleftrightarrow{AP}$. What lines are suggested by other pairs of these four points?





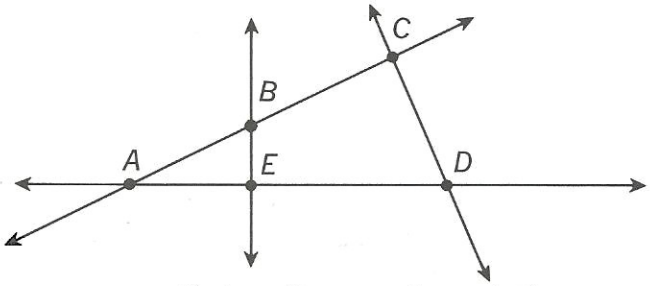
6. If two different lines intersect, is it possible that they have two points in common?
7. Does a line have an endpoint?
8. Explain what the figure suggests about the make-up of a line.
9. Draw a line m. Mark two points E and F on m.
   [a] If two people hold their drawings together, do you think their pairs of points will fit together?
   [b] Is it possible that no two drawings in your class are exactly alike?

Exercises

1. Mark two points on your paper and label them P and Q. Use your ruler to draw the line through P and Q. Mark a point A not on $\overleftrightarrow{PQ}$. Draw $\overleftrightarrow{AP}$ and $\overleftrightarrow{AQ}$.
2. If three points are chosen, not on the same line, how many lines are suggested by pairs of these points?
3. Mark four points on your paper and label them A, B, C, and D. Make sure no three of them are on the same line. Draw all the lines determined by these points. Give the names of these lines, such as $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and so on. You should have six lines in all.

★4. How many lines would you get from five points if no three of them were on the same line?

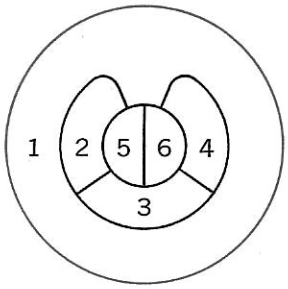
5. [a] In the figure, how many lines are shown?
   [b] Is it true that $\overleftrightarrow{AB} = \overleftrightarrow{BC}$? (Is the line indicated by $\overleftrightarrow{AB}$ the same line as the one indicated by $\overleftrightarrow{BC}$?)
   [c] Give two other names for $\overleftrightarrow{AE}$.



6. Draw four lines, m, n, o, p, so that no three go through the same point and each one intersects the other three. How many points are determined by intersections of pairs of these lines?

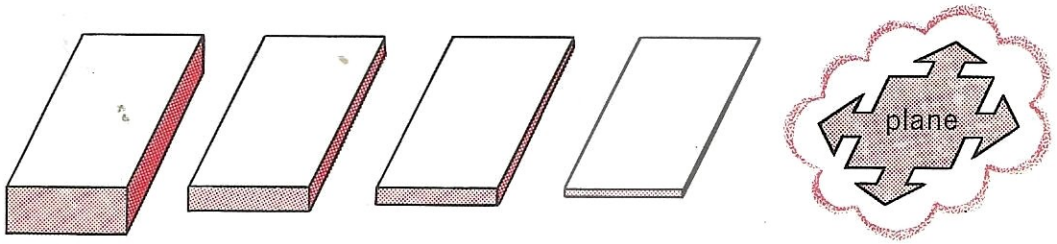
Think

Copy the figure on your paper. If you had six different coloring crayons, you could easily color the six regions of this "map" so that no two bordering regions had the same color. Try to do it using only five colors. Then try to color the map using only four colors. Can you color the regions using only three colors? Remember, bordering regions may not have the same color.



3-2 PLANES

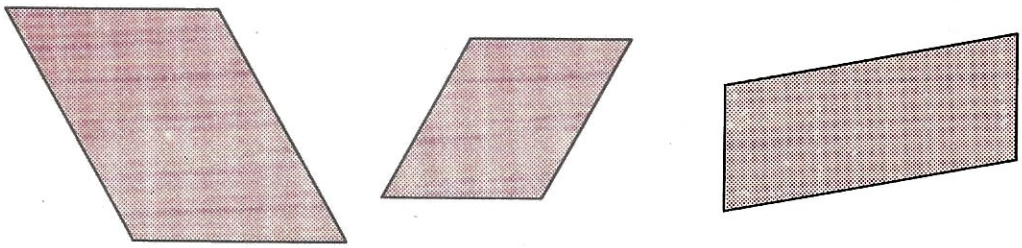
We cannot see points and lines. Neither can we see planes. A plane is a special set of points.

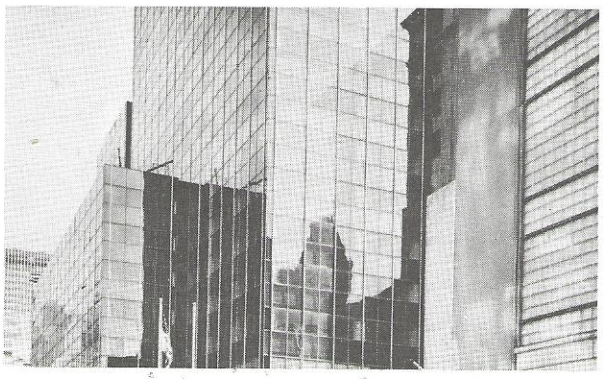


Slab Thin slab Thinner slab Thinner slab An idea

We see things, such as walls, floors, and the sides of buildings, which suggest the idea of a plane. Your desktop and the chalkboard suggest planes.

We draw pictures of planes, or rather we draw pictures of **parts** of planes. Here are some:

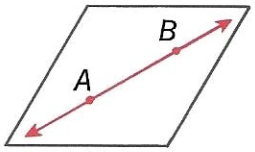




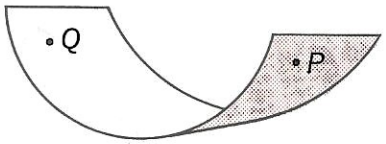
Part of the facade of the Corning Glass Works building, New York City

Discussion Exercises

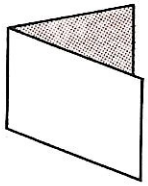
1. Look around your classroom and find objects which suggest planes.
2. In what ways does a sheet of paper fail to be like a plane?
3. Put the edge of your ruler on your desktop. The desk should touch all along the edge of the ruler. What does this suggest about a line on a plane?
4. If two points A and B are on a plane, what can you say about all the points on the line AB? The answer to this question should help you understand why we say that a plane is flat.



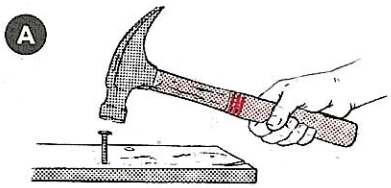
5. The figure at the right pictures a nonflat surface. Explain, using points P and Q, why it is not flat.

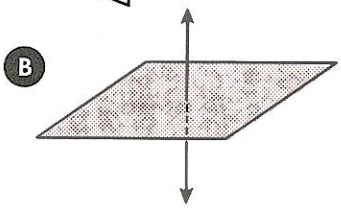


6. Carefully fold a piece of paper as in the figure. What does the crease in the paper suggest about the intersection of two planes?

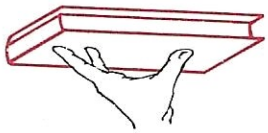


7. What do Figures A and B suggest about the intersection of a plane and a line not entirely in the plane?

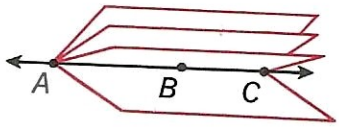




8. We say that two points determine exactly one line. What does the figure suggest about three points and a plane?

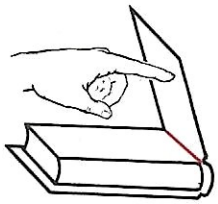


9. What does the figure suggest about the number of planes through three points that are on the same line?



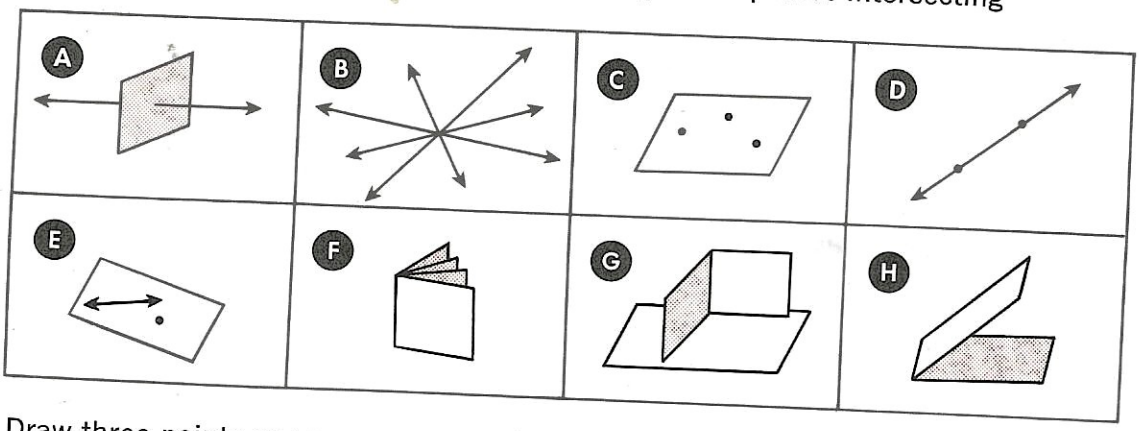
Exercises

1. What does the figure suggest about the number of planes determined by a line and point not on the line?



2. Give the correct figure for each exercise.

[a] Two points determining one line
[b] Four planes intersecting in one line
[c] A line intersecting a plane in a point
[d] A line and a point determining a plane
[e] Four lines intersecting in a point
[f] Two planes intersecting in a line
[g] Three points determining one plane
[h] Three planes intersecting

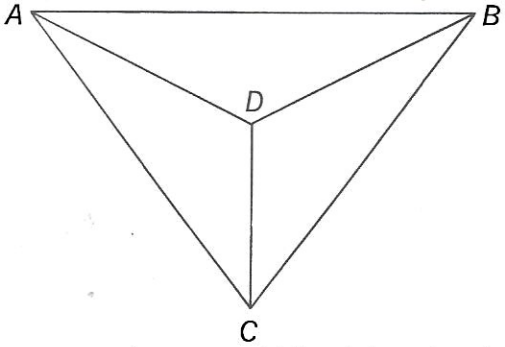


3. Draw three points on your paper so that just one plane contains these points.

4. Draw three points on your paper so that each of many planes contains these points.

5. [a] Will a four-legged chair always sit on the floor so that all four legs are on the floor?
[b] What principle of geometry helps you explain why a three-legged stool will always sit "squarely" on the floor?

6. In the figure there are four points so that no three lie on one line and no plane contains all of them. Points *A*, *C*, and *D* determine one plane called plane *ACD*. Name three other planes determined by other sets of three points in the figure.

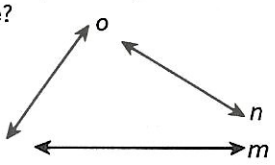


★**7.** How many planes would be determined by five points? (See Exercise 6.)

Think

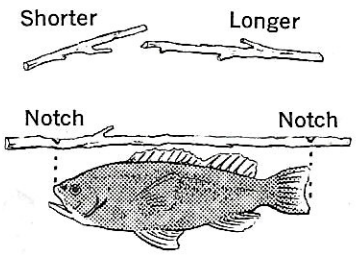
If *m*, *n*, and *o* are three different lines in space, consider the number of points of intersection of these lines.

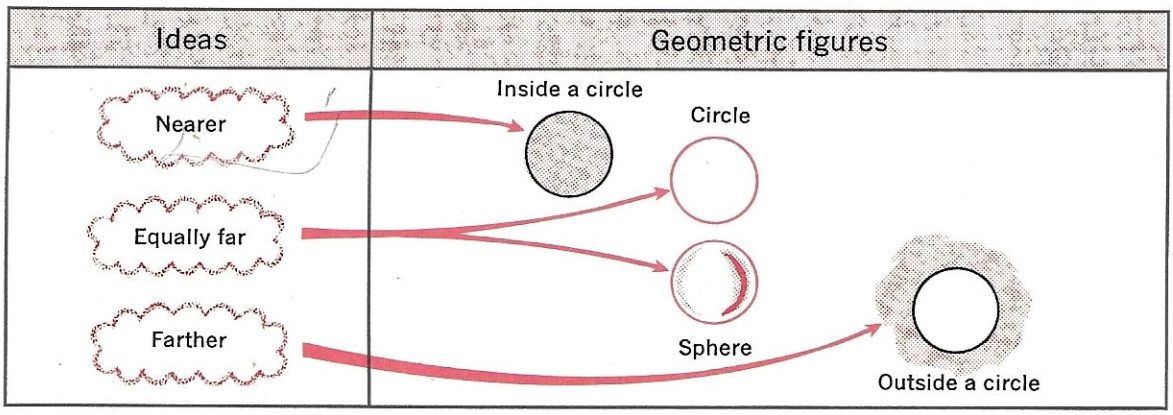
Can they intersect in no points? One point? Two points? Three points? Four points? If the lines were all in the same plane, would your answers be the same?



3-3 FARTHER, NEARER, EQUALLY FAR

We do not need to measure (get a number) to tell which of two sticks is the longer one. Two notches on a stick will preserve a record of a fisherman's skill long after the fish is eaten. Using ideas of comparison, you will see how we can describe such geometric ideas as a circle, a sphere, the inside and outside of a circle, and even the straightness of a line.

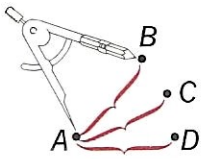




Exercises

1. Mark two points A and B on your paper. Use your compass to find a point C that is just as far from A as B is from A. Now find a fourth point D that is just as far from A as B and C are.

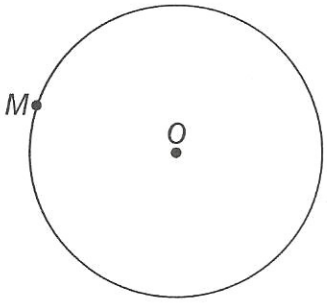
Points B, C, and D are equally far from A.



2. Mark point O on your paper. Find 10 points on your paper that are just as far from O as M is from O.

3. [a] Using the compass setting from Exercise 2, draw on your paper the set of all points that are just as far from O as M. What you have is the circle through M with center O.

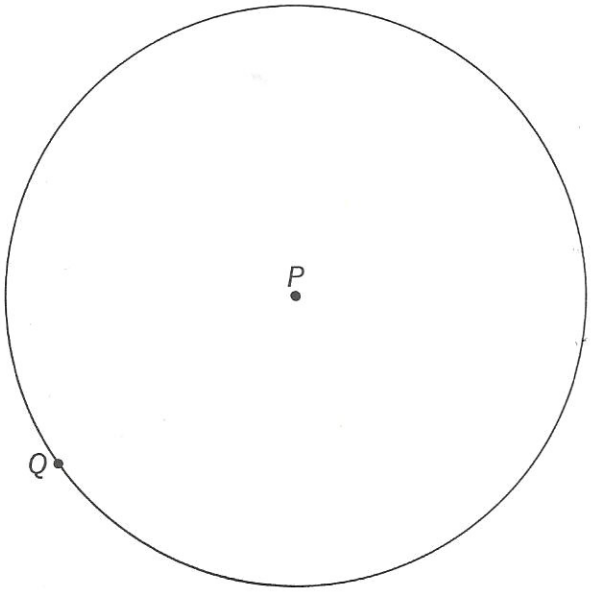
★[b] If you considered all the points in **space** (rather than just in a plane) that are as far from O as M, what kind of a geometric figure would you have?



4. Mark two points P and Q about $1\frac{1}{2}$ in. apart. Draw the circle through Q with P as center. Mark a point A that is farther from P than Q is. Mark a point B that is nearer P than Q is. Complete the following sentences with the word **inside** or **outside**.

[a] The set of all points farther than Q from P is _?_ the circle.

[b] The set of all points nearer than Q to P is _?_ the circle.



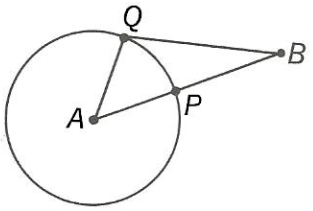
★**5.** Describe the inside and outside of a sphere as you did for a circle in Exercise 4.

6. Answer the following questions about the figure.

[a] Which path is shorter from A to B, through P or through Q?

[b] Are P and Q equally far from A?

[c] Which point, P or Q, is farther from B?

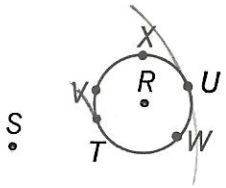


7. Point R is the center of the circle.

[a] Name several points that are equally far from R.

[b] Of these points, which one is nearest S?

[c] Which one is farthest from S?

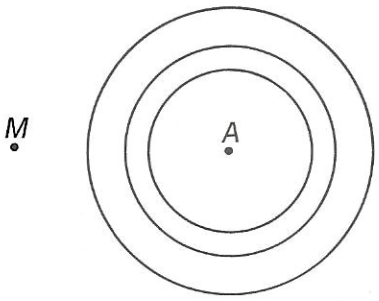


8. Draw three circles that have the same center point. Label the center A and mark a point M outside the largest circle.

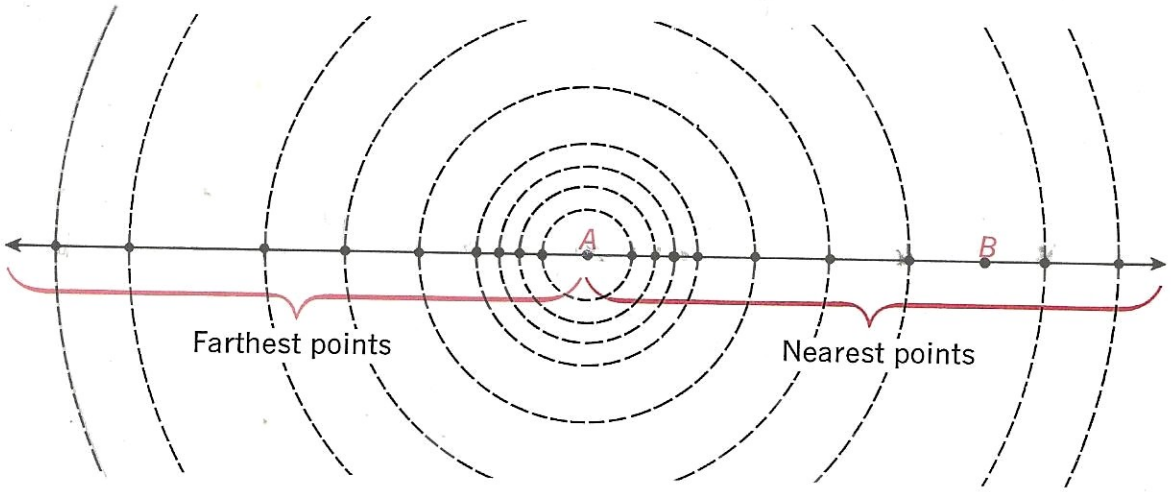
[a] Mark on the largest circle the point that is nearest M. Do the same for the other circles.

[b] Mark on each circle the point that is farthest from M.

[c] With A as center, draw another circle large enough so that M is inside it. Mark the point on this circle that is nearest M; farthest from M.

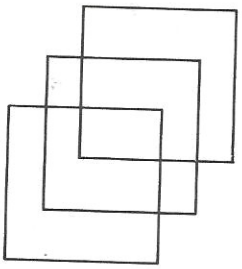


9. The figure below shows two points A and B with many circles having center A. The points nearest B and the points farthest from B are marked on each circle. What can you say about the location of all these points?



Think

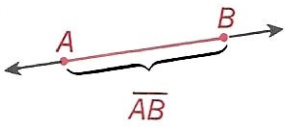
Starting at any point, draw the figure without crossing any lines, retracing any lines, or lifting your pencil from the paper.



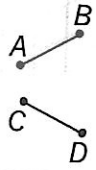
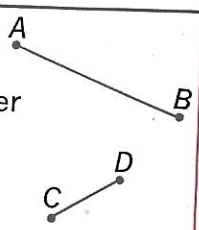
3-4 CONGRUENT SEGMENTS

Suppose you are told that three points on a line are labeled A, B, and T. You are also told that points A and B are farther apart than A and T and farther apart than B and T. Try to visualize where T is. You should see that T is **between** A and B.

The segment AB with endpoints A and B is made up of points A and B and all points between A and B.

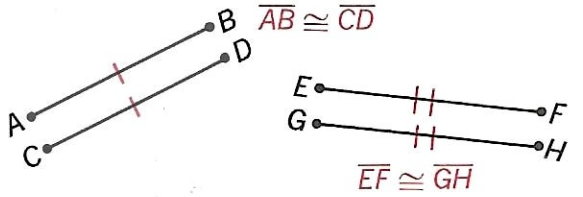


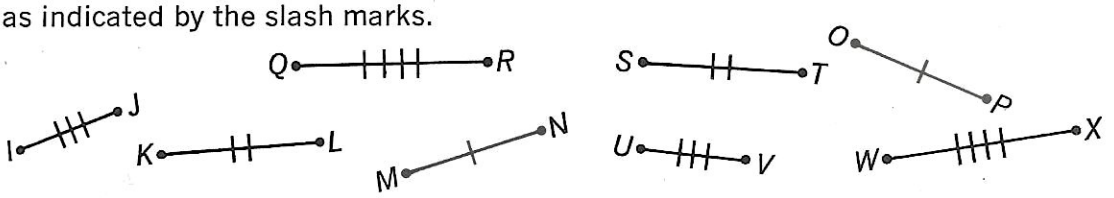
Comparing segments

If	We say	We write
two segments have their ends equally far apart, 	the segments are congruent	$\overline{AB} \cong \overline{CD}$
the ends of one segment are closer together than the ends of another, 	the first segment is greater than the second or the second is less than the first.	$\overline{AB} > \overline{CD}$ or $\overline{CD} < \overline{AB}$

Exercises

1. We often use slash marks to indicate that two segments are congruent. The same number of slashes indicates that the two segments are congruent. Find pairs of congruent segments as indicated by the slash marks.



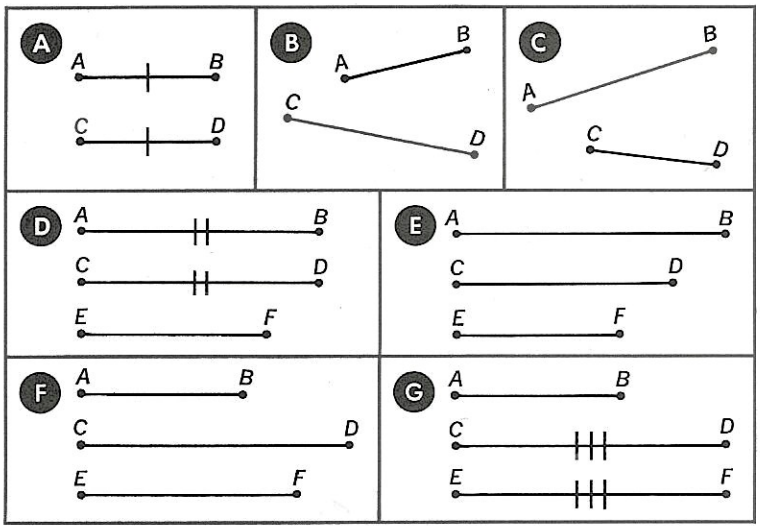


2. Select the correct figure.

[a] $\overline{AB} > \overline{CD}$
[b] $\overline{AB} \cong \overline{CD}$
[c] $\overline{CD} < \overline{AB}$
[d] $\overline{CD} > \overline{AB}$
[e] $\overline{AB} < \overline{CD}$

3. Select the correct figure.

[a] $\overline{AB} < \overline{CD}$ and $\overline{EF} < \overline{CD}$
[b] $\overline{AB} > \overline{CD}$ and $\overline{CD} > \overline{EF}$
[c] $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EF}$
[d] $\overline{CD} > \overline{EF}$ and $\overline{EF} > \overline{AB}$
[e] $\overline{AB} \cong \overline{CD}$ and $\overline{AB} > \overline{EF}$
[f] $\overline{AB} > \overline{EF}$ and $\overline{CD} < \overline{AB}$



4. Label four points A, C, E, K in that order on a line. Name all different segments. There are six in all.

5. Write three inequality statements about the six segments in Exercise 4. Example: $\overline{AC} < \overline{AE}$

6. Select the statement that you think completes the sentence correctly. Be prepared to explain your selection.

[a] If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, we know that $\begin{cases} \overline{AB} \cong \overline{EF}. \\ \overline{AB} < \overline{EF}. \end{cases}$

[b] If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} > \overline{EF}$, we know that $\begin{cases} \overline{AB} \cong \overline{EF}. \\ \overline{AB} > \overline{EF}. \end{cases}$

[c] If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, we know that $\begin{cases} \overline{AB} < \overline{EF}. \\ \overline{AB} \cong \overline{EF}. \end{cases}$

★**7.** Suppose you know that $\overline{AB} > \overline{CD}$ and $\overline{AB} > \overline{EF}$. Draw pictures to show why you cannot be sure how $\overline{CD}$ and $\overline{EF}$ compare.

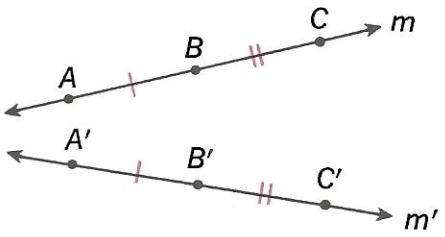
★**8.** Points A, B, C are on a line m, and B is between A and C. Points A', B', C' are on a line m' and B' is between A' and C'.

[a] If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, what can you say about $\overline{AC}$ and $\overline{A'C'}$?

[b] Suppose that B' is not between A' and C', but all the other relations hold. How do $\overline{AC}$ and $\overline{A'C'}$ compare?

Think

All four points A, B, C, and D are on one line. $\overline{AB} < \overline{BC} < \overline{CD}$, and $\overline{AD} < \overline{CD}$, and $\overline{BD} < \overline{CD}$. Draw a picture showing how this can happen.



★**9.** Points A, B, C are on a line m and B is between A and C. Points A', B', C' are on line m' and B' is between A' and C'.

[a] If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$, what can you say about $\overline{BC}$ and $\overline{B'C'}$?

[b] If B' is not between A' and C', what can you say about $\overline{BC}$ and $\overline{B'C'}$?

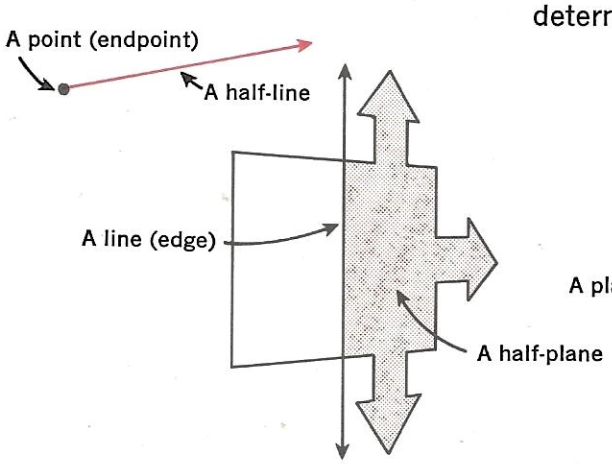
3-5 RAYS, ANGLES, AND SPECIAL SETS

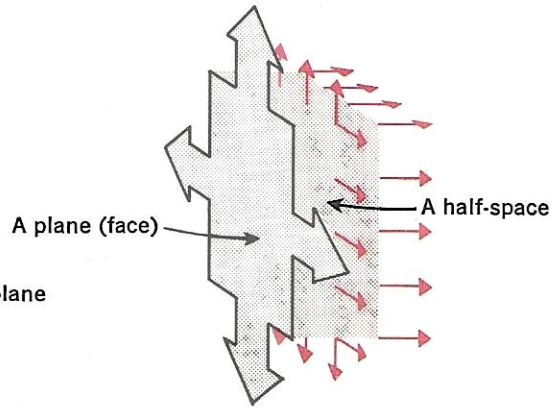
Each of the two sides of a point on a line is called a **half-line.**

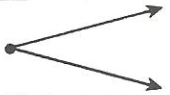
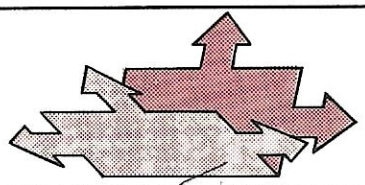
Each of the two sides of a line in a plane is called a **half-plane.**

Each of the two sides of a plane in space is called a **half-space.**

Observe that these sets do **not** contain the endpoint, the edge, or the face that determines them.

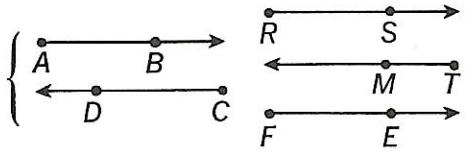




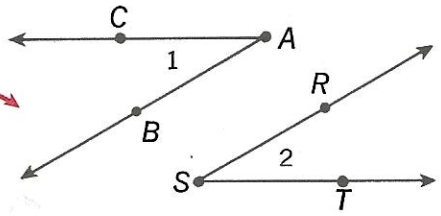
Some special geometric figures		
Description	Picture	Name
A half-line **and** the endpoint		Ray
Two rays with a common endpoint		Angle
Two half-planes with the same edge		Dihedral angle

Exercises

1. The symbols for these two rays are $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Give names for the other three rays.



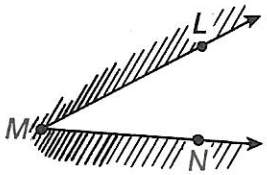
2. The three methods commonly used to name this angle are $\angle BAC$, $\angle A$, and $\angle 1$. Use these methods to name the other angle.



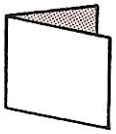
3. The two rays are called the **sides** of the angle. In Exercise 2, the sides of $\angle CAB$ are $\overrightarrow{AC}$ and $\overrightarrow{AB}$. Give the sides of $\angle RST$.

4. The **vertex** of $\angle BAC$ is the point A. What is the vertex of $\angle RST$?

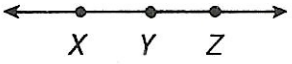
5. The **exterior** of $\angle LMN$ is shaded ////. Draw an angle and shade the region that you think is the **interior** of the angle.



6. Fold a sheet of paper as in the figure. What geometric figure does this suggest?



7. When the two different rays of an angle lie on a line, the angle is called a **straight angle**. Give the name of the straight angle with vertex Y.



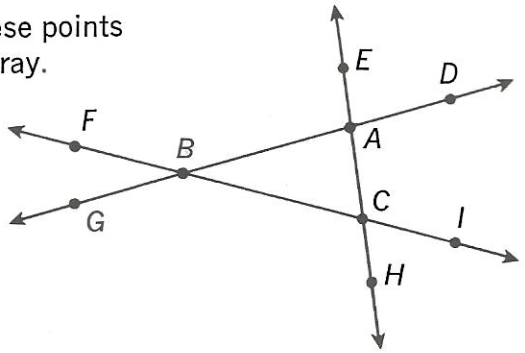
8. Draw two lines intersecting at a point P.

[a] How many rays are formed?

[b] Make a point on each ray and label these points A, B, C, and D. Give the name of each ray.

[c] There are two straight angles. How many other angles are formed?

[d] Name each of the angles.



★ **9.** [a] In the figure, name 18 different rays.

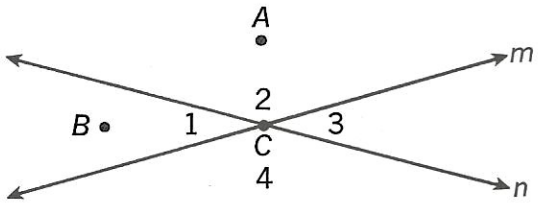
[b] Name 12 different angles, none of which is straight.

★ **10.** Draw an angle and label it $\angle DEF$. Shade with lines ////// the half-plane containing F and with edge $\overleftrightarrow{DE}$. Shade with lines \\\\\\ the half-plane containing D and with edge $\overleftrightarrow{EF}$. What is the intersection of the two shaded regions?

★ **11.** Draw two intersecting lines m and n and choose points A and B as in the figure. Shade ////// all points on the B side of m, and shade \\\\\\ all points on the A side of n.

[a] What is the intersection of these sets?

[b] What is their union?



Think

To a bug that lives on a line (a one-dimensional bug), a jail has two sides, the ends A and B of a segment.

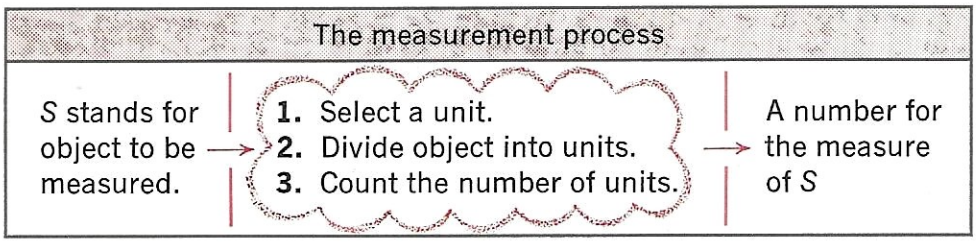
1. What is the minimum number of straight sides needed to form a jail for a two-dimensional bug?
2. What is the minimum number of flat sides needed to form a jail for a real (three-dimensional) bug?

B. METRIC GEOMETRY

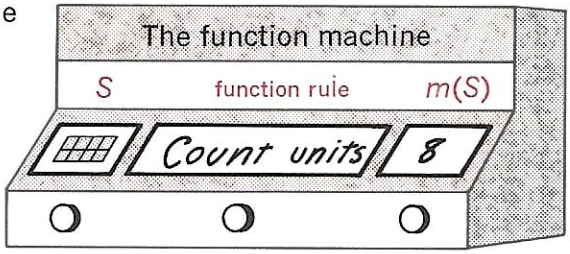
We have studied geometric figures in nonmetric geometry. Now we can begin to measure these figures.

3-6 THE MEASUREMENT PROCESS

When we measure a set or object, we assign a number to it. Much of your work with measurement will be concerned with measuring various objects.


The measurement process
S stands for object to be measured.
1. Select a unit.
2. Divide object into units.
3. Count the number of units.
A number for the measure of S

Thinking about the function machine may help you better understand measurement. You can think of the **output**, $m(S)$, as the number for the measure of the **input**, S.



Discussion Exercises

1. Name several units for measuring length. Compare the size of these units. If one unit is larger than another and you are measuring the same object, which unit gives the larger number?
2. Name several units for measuring volume. Compare the size of these units. If one unit is larger than another and you are measuring the same object, which unit gives the larger number?
3. Name some units for measuring time. Is the **month** a precise unit?
4. Why do you prefer to have your car odometer read miles per hour rather than miles per week or miles per second?
5. Name some units for measuring weight. How do they compare in size?

Exercises

1. Using the units suggested in the table, tell which unit you would use to find the following. In each case, tell whether you would be finding length, area, or volume.

[a] The height of a flag pole
[b] The space inside a cardboard carton
[c] The size of your garden
[d] The distance for a foot race
[e] The amount of floor space in a house
[f] The amount of concrete needed for a sidewalk

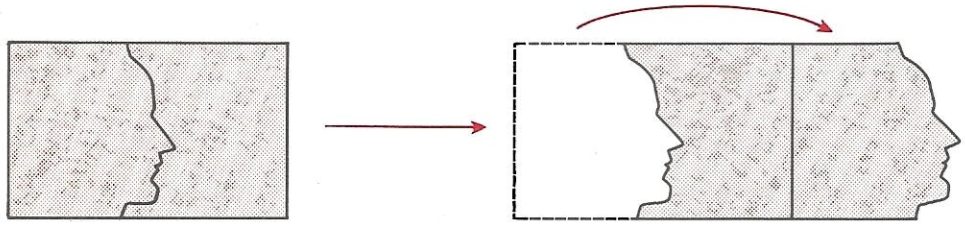
Idea	Unit
Length	Segment
Area	Square
Volume	Cube

2. Tell which idea (length, area, or volume) is being discussed.

[a] John claimed that he had painted the entire inside of the closet.
[b] John then said that he had completely filled the closet with boxes of clothes.
[c] The closet was wide enough to accommodate three boxes.
[d] He also noted that he had used two cans of paint for the job.
[e] In putting the cartons into the closet, he figured he had to walk back and forth across the room 115 times.
[f] John's father noted that John had forgotten to paint the closet door.

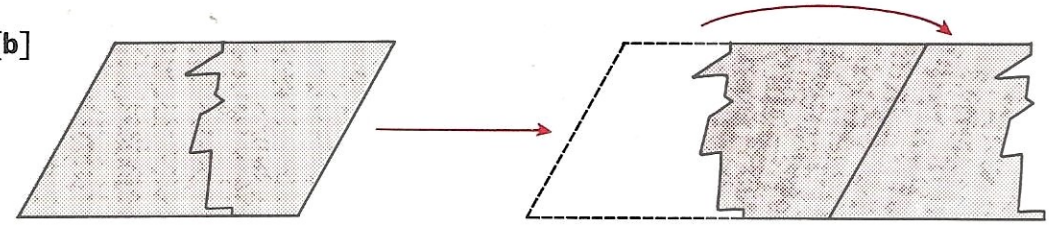
3. The measure of each rectangle is given. In each exercise, give the measure of the oddly shaped colored region.

[a]



Measure = 32 Measure = ?

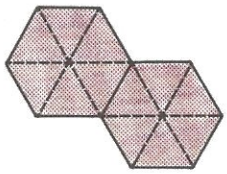
[b]

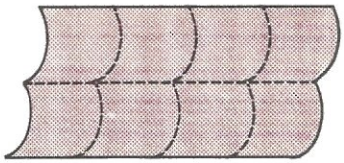


Measure = 35 Measure = ?

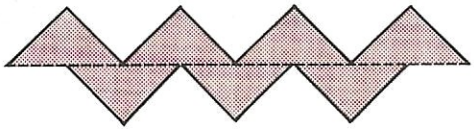
4. Study each figure. Then draw what you think would be a convenient unit to use for measuring the object.

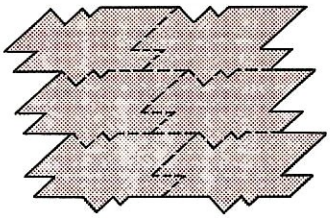
[a]

[b] 

[c] 

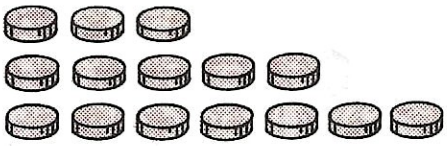
[d]

[e] 

[f] 

Think

The game Nim is played by two persons using three rows of counters, 3 in the first row, 5 in the second, and 7 in the third. Each player in turn takes one or more counters from just one row. The player who is forced to remove the last of the 15 counters loses the game.



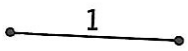
For example, the first player might take all 7 counters in the third row, and the second might remove 2 from the second row. This leaves two rows of 3.

1. Show that now the second player can win.
2. Show that if you can play and leave a 1, 2, 3 combination (that is, 1 counter in one row, 2 in another, and 3 in the third), you can win.
3. Show that if you can leave a 1, 4, 5 combination, you can win.
4. Who can win at Nim, the first player or the second?

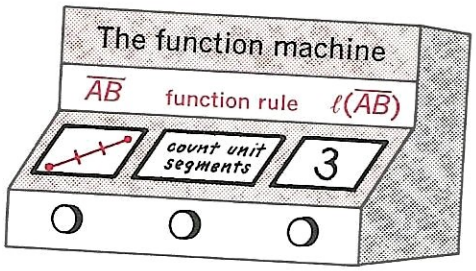
3-7 LENGTH

Consider the idea of **length**. We often use other words like **height** and **distance** to describe this idea. We speak of the length of a piece of string, the distance between New York and Chicago, and the height of a flagpole; but the same idea is present in all these cases. By this we mean that we choose a unit of measure and then give a number which tells approximately how many of these units it would take, lying end to end along a straight path, to reach from one point to another.

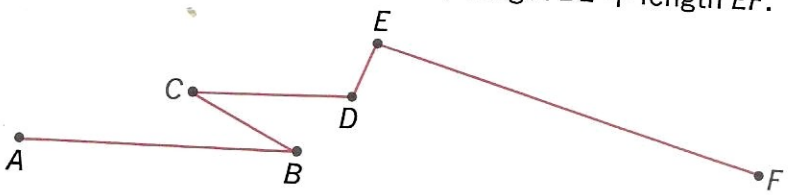
The length of a segment is a number that compares its size with the size of a unit segment.



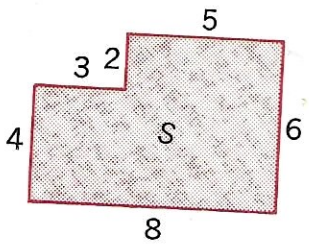
We call the length the **measure** of the segment.



The **polygonal path** *ABCDEF* is the union of the segments *AB*, *BC*, and so on. Its length is length $\overline{AB}$ + length $\overline{BC}$ + length $\overline{CD}$ + length $\overline{DE}$ + length $\overline{EF}$.



If region *S* has a boundary which is a polygonal path, then the **perimeter** of *S* is the length of the path.



Perimeter *S* = 28

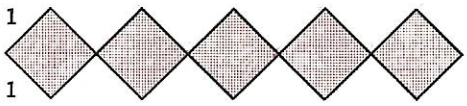
Discussion Exercises

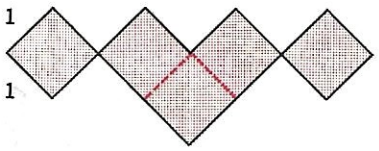
1. Describe how you would measure the width of your room with a foot ruler.
2. Describe the measurement involved when a football official marks off a 15-yd penalty. What is his unit? Is counting involved?
3. Measure the width of the top of your desk using the width of your hand as a unit. Is this an accurate measurement? Was your result a whole number?

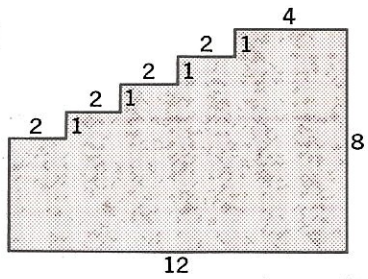
4. Suppose you are given an unmarked stick to use as a unit in measuring the length of a table. You could probably find the length to the nearest whole number of "sticks." What mark would you need on the stick to be reasonably sure that your measurement was correct to the nearest unit?
5. What marks would you need on a measuring stick if you desired to measure to the nearest half of a unit? to the nearest fourth of a unit?
6. Name as many units as you can which are in common use for measuring length.
7. Suppose the perimeter of a rectangle is 16 in. What can you say about its area? What is its greatest possible area?
8. Suppose someone says, "The length of this table is 24 units." After measuring it with a different unit, you find the length to be 6. Which is the larger unit, the first or the second? How much larger?
9. Explain why the measure of something, obtained using the foot as a unit, is approximately three times the measure obtained using the yard as a unit. What is the relationship between measures obtained by the foot and by the inch?

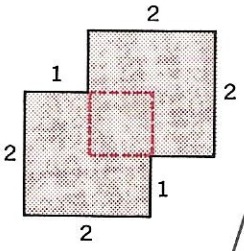
Exercises

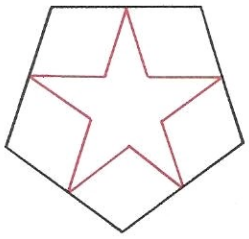
1. What is the perimeter of each region?

[a] 

[b] 

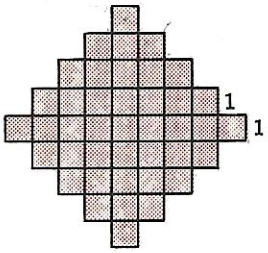
[c] 

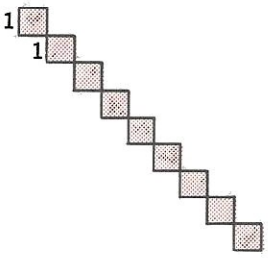
[d] 

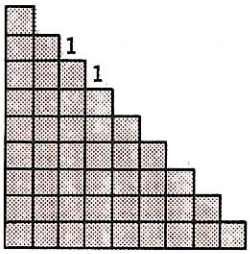


2. How can you arrange nine unit squares, nonoverlapping, to form a region with the least perimeter? The greatest perimeter?
3. How do the perimeters of the colored star and the black figure compare?

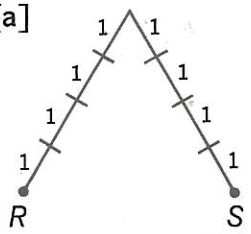
4. What is the perimeter of each region?

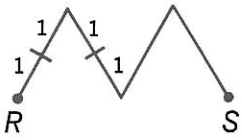
[a] 

[b] 

[c] 

5. What are the lengths of the following polygonal paths between R and S?

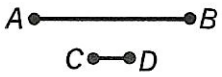
[a] 

[b] 

[c]

[d]

6. The measure of $\overline{AB}$ in the figure is 5 when $\overline{CD}$ is used as a unit. If the width of a desk is 18 when $\overline{AB}$ is used as a unit, what is the width of the desk when $\overline{CD}$ is used as a unit?



★**7.** In measuring the distance between two trees, using a stick as your unit, you find it is about $8\frac{1}{2}$ "sticks." You now measure the stick and find that it is about $3\frac{1}{2}$ ft. What is the approximate distance, in inches, between the trees?

★**8.** Consider two measuring units, $\overline{AB}$ and $\overline{CD}$. The measure of $\overline{AB}$ with $\overline{CD}$ used as a unit is 3.

[a] What is the measure of $\overline{CD}$ with $\overline{AB}$ used as a unit?

[b] If the measure of something is x when $\overline{AB}$ is used as a unit, what is its measure with $\overline{CD}$ as a unit?

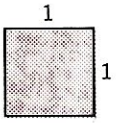
★**9.** Suppose two measuring units, $\overline{AB}$ and $\overline{CD}$, are related so that the length of $\overline{AB}$, with $\overline{CD}$ used as a unit, is r. If the length of an object is t when $\overline{AB}$ is used as unit, what is its length when $\overline{CD}$ is used as the unit?

Think

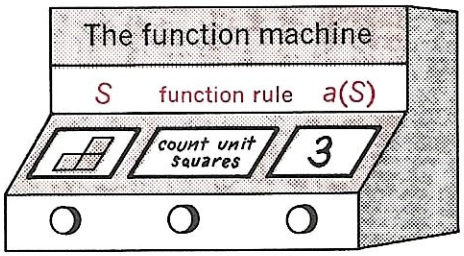
You have two pails, one that will hold 4 quarts (qt) and one that will hold 9. There are no markings on either pail to indicate smaller quantities. How can you measure out 6 qt of water, using these two pails?

3-8 AREA AND SURFACE AREA

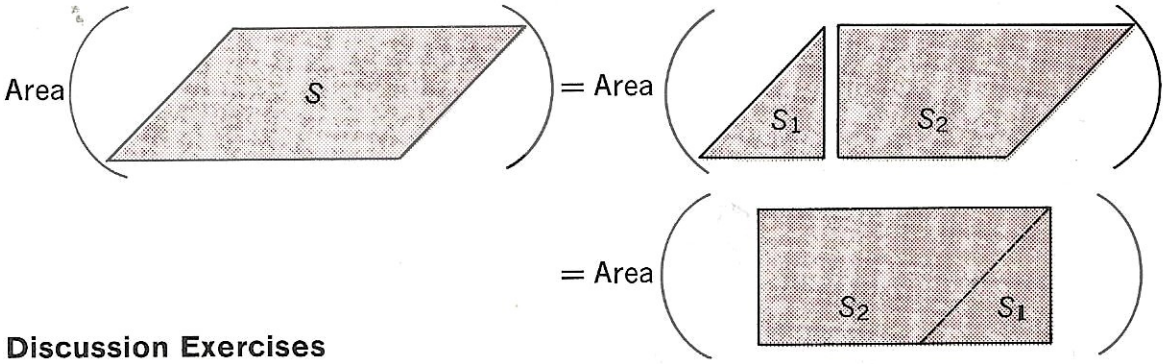
As with the concept of length, we must choose a unit and count to find the **area** of a given region. We usually choose a unit of length and then use as the "area unit" a square region that has sides of length 1. Then we count to see how many of these **unit-square** regions it takes to "fill up" or "cover" some region S. The number of units needed is the area of S.



Unit square

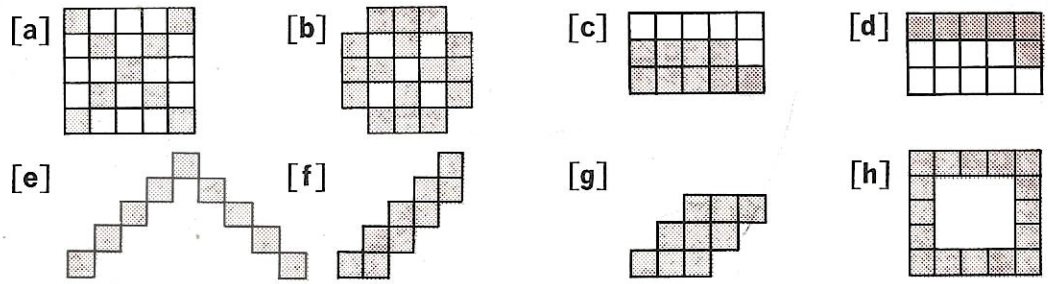


Note that a rectangle is made of four segments. When we say area of a rectangle, we mean the **area of the interior** of the rectangle. We sometimes speak of the area of a rectangular **region**. Sometimes it is easier to see how a figure can be measured by square regions if we "cut it apart."

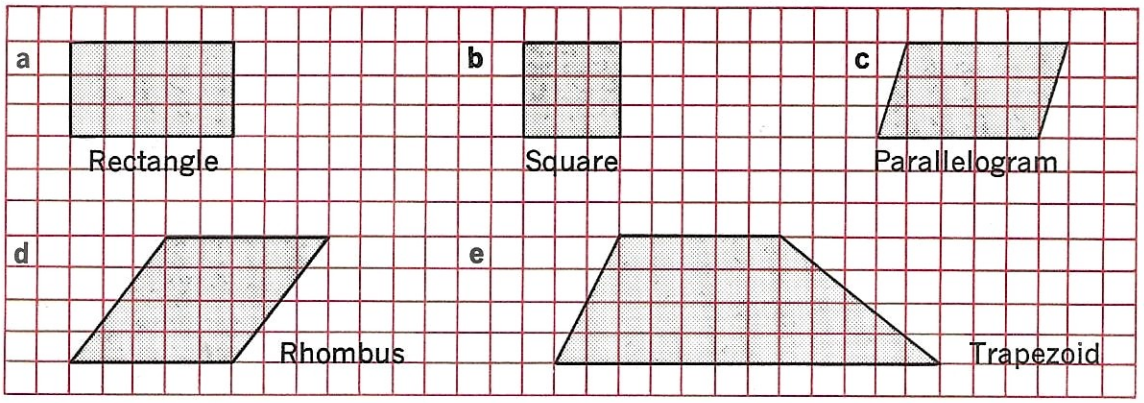


Discussion Exercises

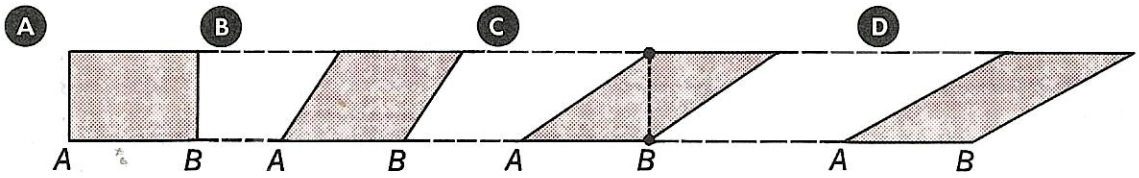
1. All squares shown are unit squares. Find the area of the colored regions.



2. Estimate the area for each geometric figure. Unit squares are indicated by the colored grid.

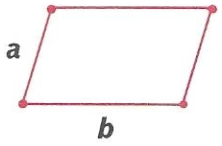


3. How do you think the areas of the following figures compare with each other? All the segments $\overline{AB}$ are congruent.



★**4.** Suppose a parallelogram is made of rods hinged at the corners.

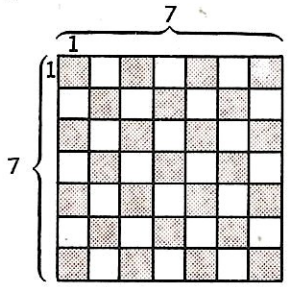
[a] Is the figure rigid?
[b] What is the largest area it could have? Why?
[c] What is the smallest area it could have?
[d] Does the perimeter change as the hinges move?



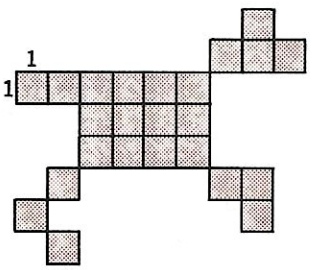
Exercises

1. In each figure, estimate the area of the colored region. Check your estimate.

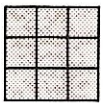
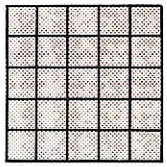
[a]



[b]

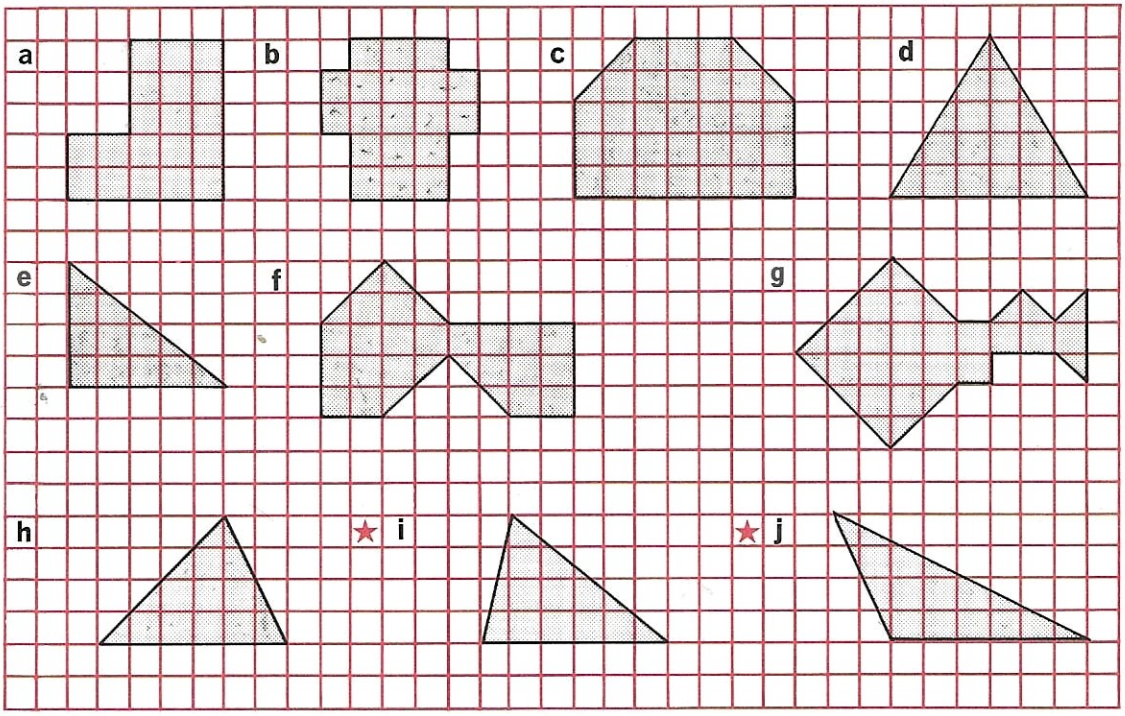


2. Is the following statement true? Why?

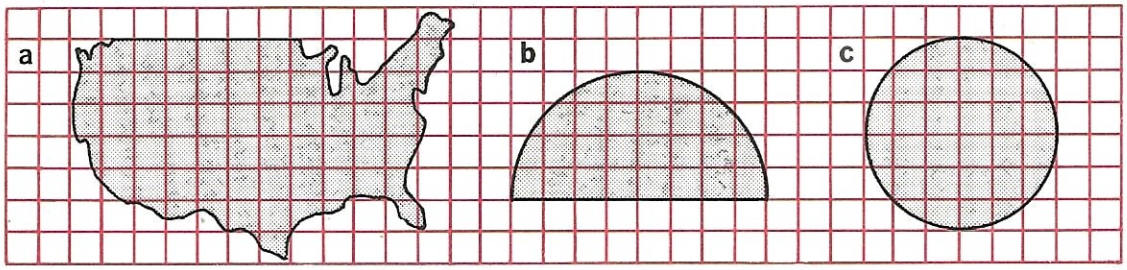
Area  + Area = Area 

3. Draw a unit square. Draw a rectangle 4 units long and 5 units wide. What are the area and perimeter of the rectangle.

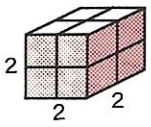
4. By counting unit squares, find the area of each shaded region.



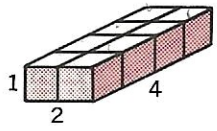
5. Estimate the area.

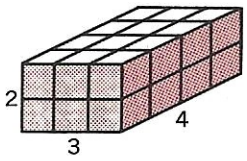


6. [a] What is the area of the front face of the cube?
[b] What is the total area of all the faces that you can see in the figure?
[c] What is the **surface area** (total area of all six faces) of the cube?



7. Find the surface area of each figure.

[a]


[b]


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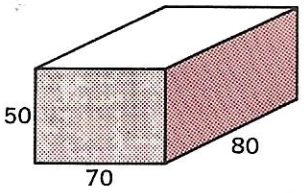
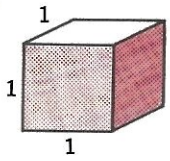
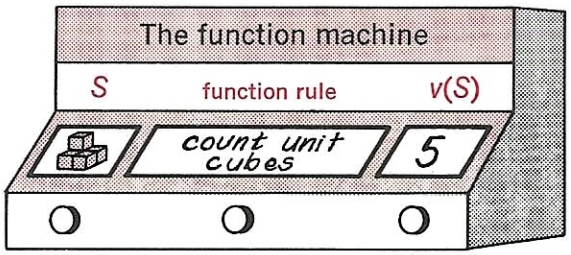


3-9 VOLUME

When we measure a region of a plane, we find area. When we measure a region of space, we find **volume**. As with area, we choose a unit and count. A common unit for volume is that region of space in a cube that has edges of length 1, a **unit cube**.

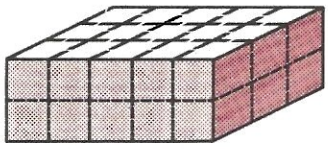


Unit cube

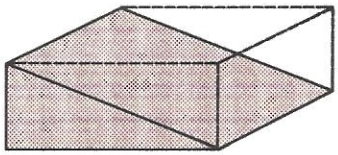


For some figures, you can easily visualize the unit cubes to be counted. For other figures, such as the wedge, you may need to use your imagination.

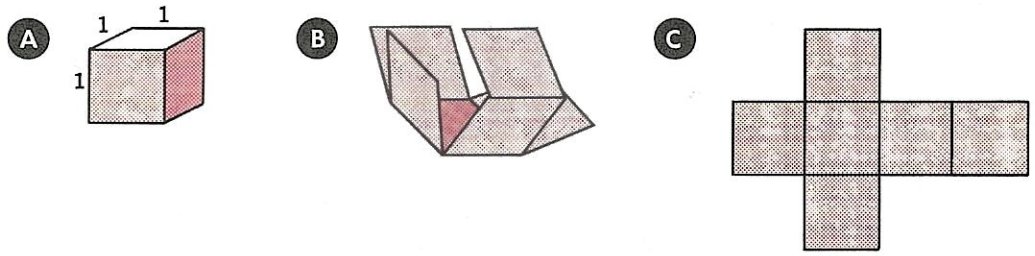
For example, the volume of this wedge is half the volume of the box shown by the dotted lines.



Volume $= 2 \cdot 3 \cdot 5$
$= 30$

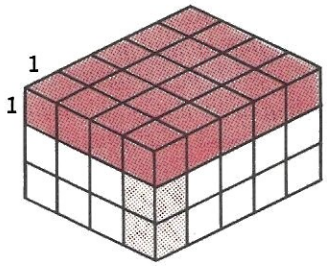


It is interesting to compare volume and **surface area**. For example, a cube with volume 1 has a surface area of 6. The following figures illustrate this.



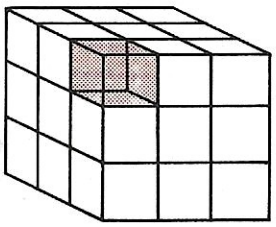
Discussion Exercises

1. [a] Discuss how you count the number of unit cubes in the solid.
[b] How many cubes in the top layer? How many cubes in all?
[c] How many cubes in the front layer? How many cubes in all?

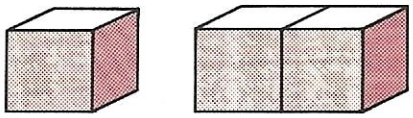


2. [a] Can you find a box whose length, width, and height are whole numbers and whose volume is 24?
[b] What surface area might such a box have?

3. [a] When the colored cube is removed, the volume is decreased by 1 unit, but the surface area is unchanged. Explain this fact.
[b] What other blocks can you remove from the figure and keep the same surface area?



4. [a] A unit cube has how many faces?
[b] When two cubes are placed together as shown, how many faces are exposed?
[c] If three cubes are placed together, how many faces are exposed?

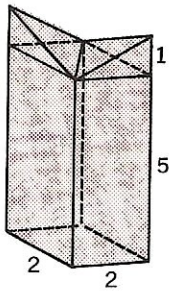


★**5.** If four cubes are placed together, how many faces
[a] **must** be exposed? [b] **can** be exposed?

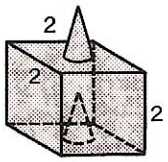
★**6.** Eight unit blocks are to be joined face to face somehow. How could they be placed so that
[a] the least surface is exposed? [b] the most surface is exposed?

★**7.** Estimate the volume of each solid. Explain your reasoning.

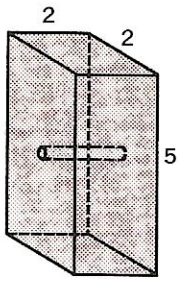
[a]



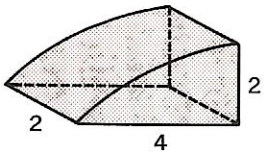
[b]



[c]



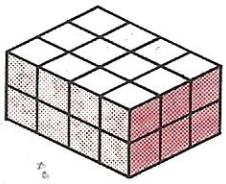
[d]



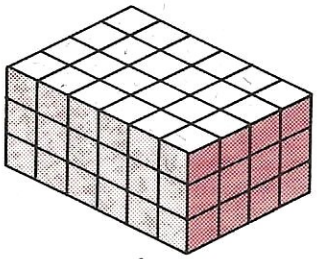
Exercises

1. Find the volume of each figure. The unit is suggested by the black lines.

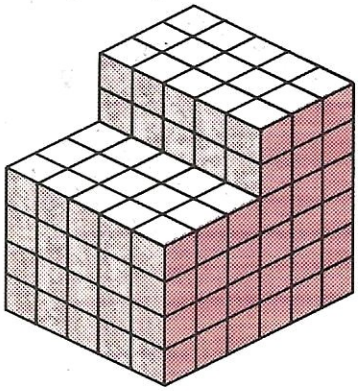
[a]



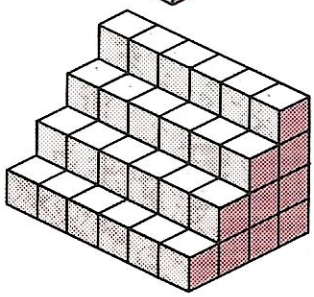
[b]



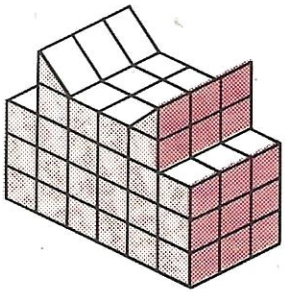
[c]



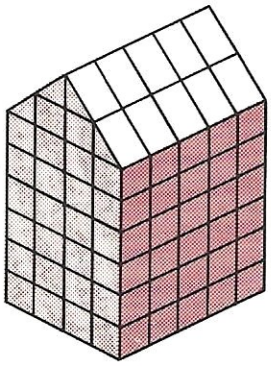
[d]



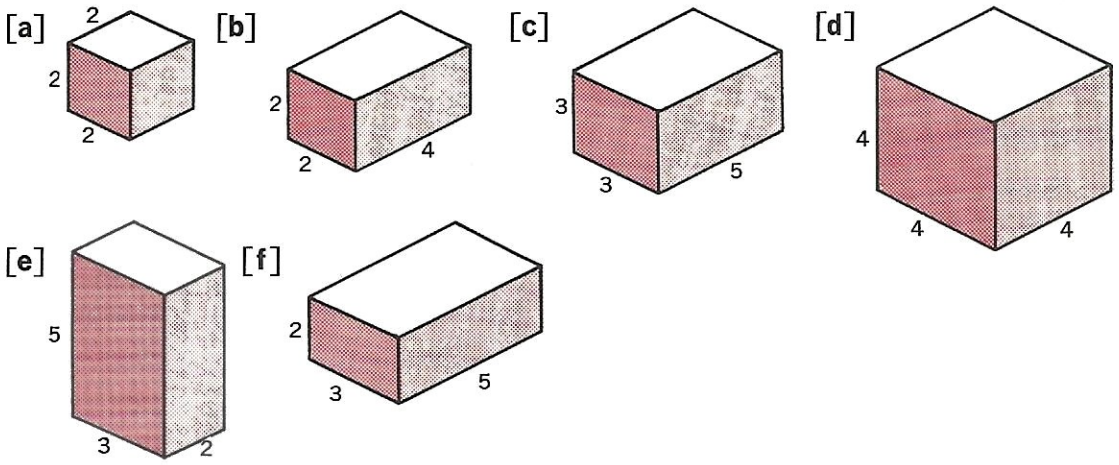
[e]



[f]

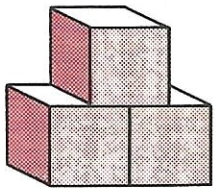


2. Find the volume and surface area of each box. The lengths of the sides are shown.



★**3.** [a] If each dimension of the boxes in Exercise 2 is doubled, how are the surface areas changed?
[b] If each dimension is tripled, how are the surface areas changed?
[c] If each dimension is doubled, how are the volumes changed?
[d] If each dimension is tripled, how are the volumes changed?

★**4.** [a] How many faces has a box?
[b] If all the dimensions are whole numbers, why must the surface area be an even number?
[c] Must the volume also be an even number?



★**5.** What is the surface area of the solid shown? (The cubes are the same size.)

CHAPTER REVIEW

Key Concepts

1. A **point** is an idea. We think of properties that we want points to have.
2. **Space** is the set of all points.
3. **Lines** are special (straight) sets of points.
4. **Planes** are special (flat) sets of points.
5. Through two points there is exactly one line.
6. Through three points, not on a line, there is exactly one plane.
7. If two points of a line are in a plane, then the whole line is in the plane.
8. Straightness can be described in terms of farthest, nearest, and equally far points.
9. **Segments** are special parts of lines.
10. **Congruent segments** have their endpoints equally far apart.
11. **Half-lines** are special parts of lines. **Half-planes** are special parts of planes. **Half-spaces** are special parts of space.

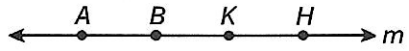
12. A half-line together with the endpoint is a **ray**.
13. To measure length, we count unit segments.
14. To measure area, we count unit squares.
15. To measure volume, we count unit cubes.

Vocabulary List

angle	greater than	less than	ray
area	half-line	line	segment
congruent segments	half-plane	metric geometry	space
distance	half-space	nearer	unit cube, segment, and square
equally far	height	nonmetric geometry	vertex
exterior	interior	plane	volume
farther	length	point	

Review Exercises

1. Draw a picture of
   [a] a point on a line
   [b] two lines through a point
   [c] a line and points on it
   [d] three rays from the same point
   [e] two rays that do not intersect
   [f] two segments that do not intersect and are parts of two lines that do intersect
2. Points E, Q, T are on a line, and $\overline{ET} < \overline{QT} < \overline{EQ}$. Draw a picture to show this.
3. Give three names for the line m.



4. Draw two lines m and n, intersecting in a point R. Mark points A and B on m so that R is between them. Mark points D and E on n so that R is between them.
   [a] Shade //// the half-plane which has edge m and which also contains D.
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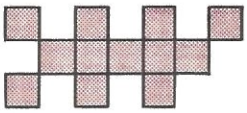
5. Points G, H, and K are on a line so that $\overline{GH} \cong \overline{KG}$. Draw a picture to show this.
6. What is an **angle**?
7. Draw a figure whose area is 12 square units. What is its perimeter? Can you get several answers?
8. [a] What is the area and perimeter of this figure?
   [b] If the squares are rearranged, what is the least perimeter you can get?



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   [a] Without counting, guess which tower has the greater surface area.
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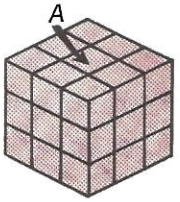
10. [a] What is the volume and surface area of the box of unit cubes?
    [b] If the cube marked A and the ones below it are removed, what is the volume and surface area of the remaining solid?



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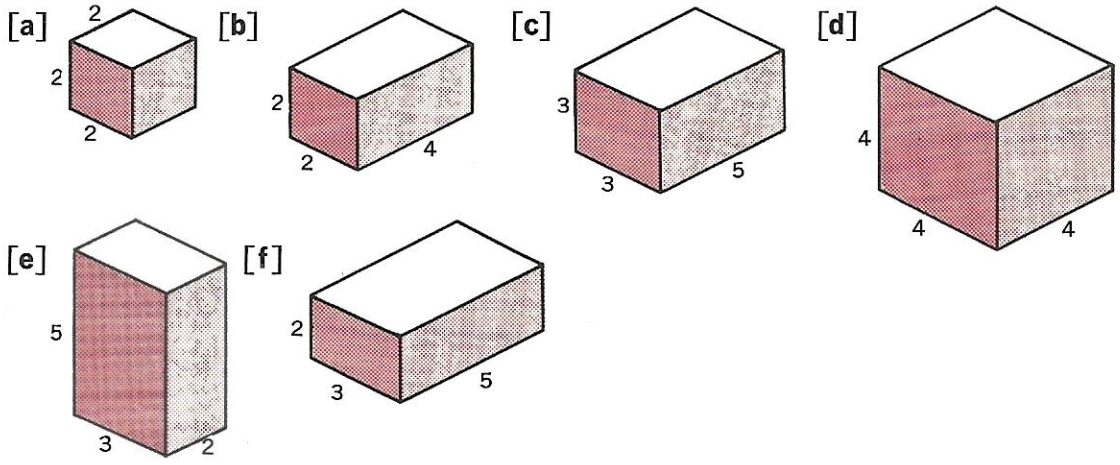

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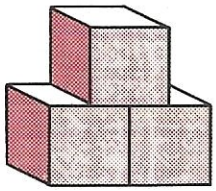

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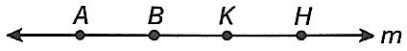

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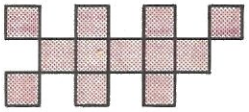

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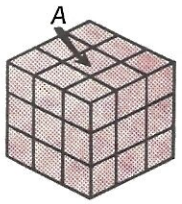
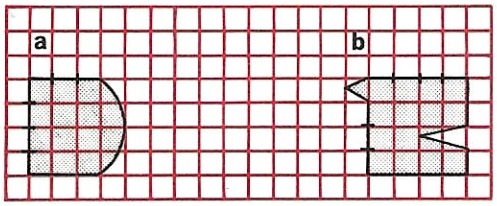
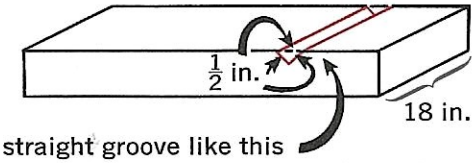


11. Estimate the areas of the following figures.



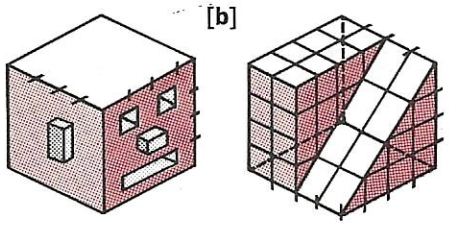
12.



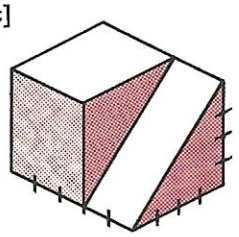
A straight groove like this is cut into the face of a block 18 in. long. By how much is the surface area of the block increased?

13. Estimate the volume of the following figures.

[a] [b]



[c]



CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 48–49.

BERGAMINI, DAVID, and the Editors of *Life. Mathematics.* Life Science Library; New York, Time, Inc., 1963; pp. 88–103.

HOGBEN, LANCELOT. *The Wonderful World of Mathematics.* Garden City, New York, Doubleday and Company, 1955; pp. 12-19.

JOHNSON, DONOVAN. *Curves in Space.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1963; pp. 1–9.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Invitation to Mathematics. Ibid.*, 1960, pp. 29–33.

———. *The World of Measurement. Ibid.*, 1961, pp. 1–26.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION. *What's Up There.* Washington, D.C., U.S. Government Printing Office, 1964; pp. 1–25.

RAVIELLI, ANTHONY. *An Adventure in Geometry.* New York, Viking Press, 1957; pp. 10–41.

SMITH, DAVID E. *History of Mathematics.* New York, Dover Publications, Vol. II, 1953; pp. 270–279, 344–368.

Activities

A. Using sticks and cardboard, construct some models of intersecting lines and planes.

B. Using magazines and newspapers, make a display of a variety of geometric figures as they occur in architecture, nature, science, advertising, and other fields.

C. Describe some objects that can be measured using ideas other than length, area, or volume. For example, discuss the units used to measure such things as time, electricity, air pressure, sound, light, and temperature.

D. Read about the metric system of measurement. What countries do not use the metric system? How are the various basic units related (for example, centimeter to millimeter)? What is a major advantage of the metric system?

E. How are units in the metric system of measurement related to the basic units in the English system?

F. Make an exhibit on measurement. Show or describe several objects that can be measured using length. Do the same for area and volume.

G. Construct a ruler by making up your own unit of measure for length and giving the unit a special name. Then compare this ruler with various standard rulers, such as the inch ruler and the centimeter ruler.

CUMULATIVE REVIEW

1. Write a digit for the number of the set of baseball players on your school's starting team.

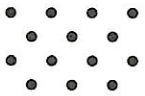
2. Give the regular numeral for each of the following:
[a] $(8 \cdot 100) + (5 \cdot 10) + 2$
[b] $(2 \cdot 10^3) + (5 \cdot 10^2) + (9 \cdot 10) + 4$

3. The numeral 10^3 means that 10 is used as a factor ▒ times.

4. Write each of the following numbers as a power of ten.
[a] 10,000,000 [b] 10,000 [c] 1

5. Write the numeral for the number that is just 11 more than 5309.

6. Consider the set of dots. Give the different base numerals which represent this set.



[a] Base ten [c] Base four
[b] Base eight [d] Base two

7. Write the Roman numeral for each of the following:
[a] 13 [b] 26 [c] 99

8. Find two numbers r and s so that $r + s = 17$ and $r - s = 9$.

9. Solve the equations.
[a] $7 + r = 15$
[b] $s - 8 = 7$
[c] $6 + (10 - 6) = n$
[d] $m \cdot 7 = 21$
[e] $(1 \cdot c) + 5 = 14$
[f] $(3 \cdot t) + 4 = 4$

10. Copy and complete the tables.

[a]

function rule $5 \cdot n$	
n	$f(n)$
2	10
3	▒
▒	25
0	▒
▒	75

[b]

function rule $(b \cdot 2) + 3$	
b	$f(b)$
0	▒
1	▒
4	▒
▒	9
▒	21

11. Find two numbers a and b so that $a \cdot b = 36$ and $a \div b = 4$.

12. Graph the set of number pairs in each of Exercises 10a and 10b.

13. Solve the equations.
[a] $17 + (47 + 83) = n$
[b] $(15 \div 15) \cdot 15 = n$
[c] $2 \cdot (19 \cdot 5) = n$
[d] $(7 \div 1) \cdot (0 \div 7) = n$
[e] $(6 \cdot 17) + (4 \cdot 17) = n$
[f] $(5 \cdot 13 \cdot 2) + (0 \cdot 14) = n$

14. Write the corresponding multiplication equation and solve.
[a] $56 \div n = 7$ [b] $n \div 12 = 6$

15. Write the corresponding addition equation and solve.
[a] $73 - n = 62$ [b] $n - 41 = 39$

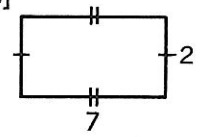
16. Choose the proper estimate for each of the following:

[a] $68 + 22 + 59$: 180, 150, 130
[b] $7823 - 2987$: 5000, 5500, 6000

17. Find the perimeter of the figures.

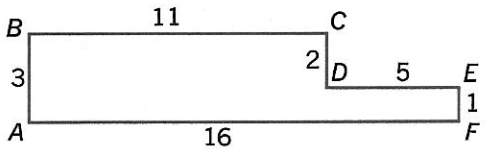
[a]

[b]

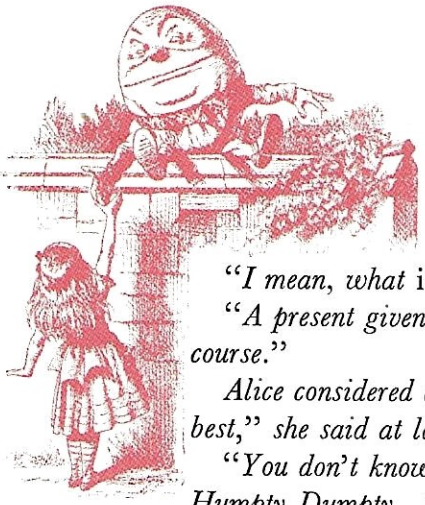


18. Which letter represents the vertex of $\angle RST$? Give another name for $\angle RST$, using only one letter.

19. Find the area of *ABCDEF*.



20. Draw two different lines, $\overleftrightarrow{PR}$ and $\overleftrightarrow{PS}$, intersecting at point *P*.



"I mean, what is *an un-birthday present?"*

"A present given when it isn't your birthday, of course."

Alice considered a little. "I like birthday presents best," she said at last.

"You don't know what you're talking about!" cried Humpty Dumpty. "How many days are there in a year?"

"Three hundred and sixty-five," said Alice.

"And how many birthdays have you?"

"One."

"And if you take one from three hundred and sixty-five, what remains?"

"Three hundred and sixty-four, of course."

Humpty Dumpty looked doubtful. "I'd rather see that done on paper," he said.

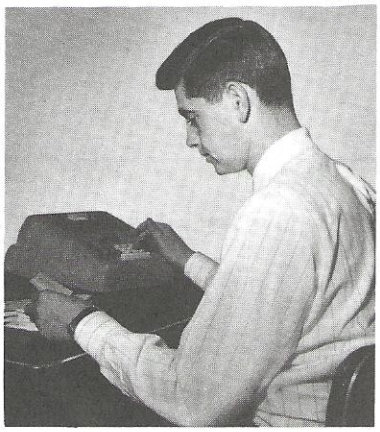
4

Computing and Problem Solving

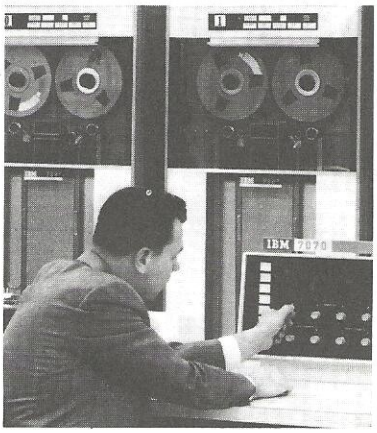
4-1 FINDING SUMS

Slow

Fairly fast



Extremely fast



Electronic digital computers are being rapidly improved. Their speeds are extremely fast. For example, the time for this addition is 10 microseconds, that is, 10 one-millionths of a second! Imagine—a hundred thousand additions in one second!

$$\begin{array}{r} 3{,}456{,}789{,}243 \\ +\ 5{,}443{,}278{,}654 \\ \hline 8{,}900{,}067{,}897 \end{array}$$

Our civilization is becoming more complex, and machines are taking over many of the simple tasks that in the past men have performed. Today, most computations in business and scientific work are done by machines. But men must build these machines, and men must prepare the programs that tell the machines what they are to do. Man's job is to **reason** and to **understand**. We turn over to machines those tasks that require no thought.

But all of us need some skill in arithmetic. It is not convenient to carry a calculating machine everywhere we go. Our brains are finer and more complicated machines than man will ever build. It is important to train them to do the problems of everyday arithmetic.

Discussion Exercises

In Examples A and B, some of the digits are covered by screens.

1. Explain why the tens digit in the sum for Example A can be either 6 or 7.

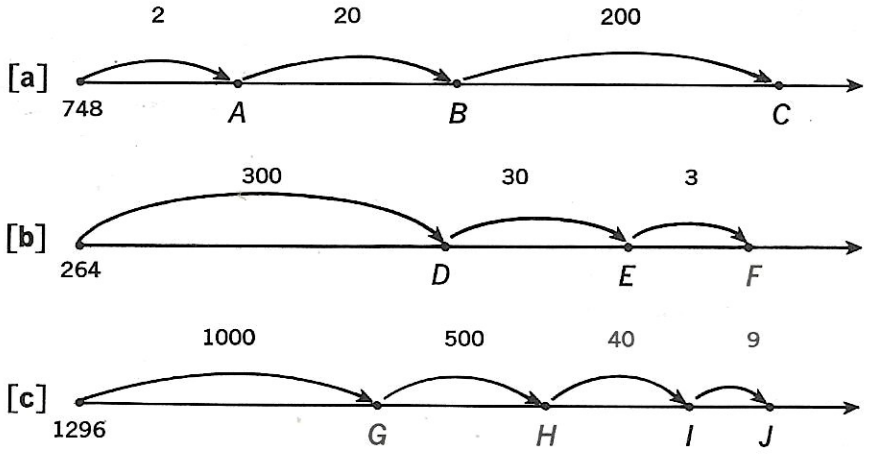
(A)
$$\begin{array}{r} \square 7 \square \\ +\ \square 9 \square \\ \hline \square\square \square \end{array}$$

2. Explain why the hundreds digit in the sum for Example B can be 1, 2, or 3.

(B)
$$\begin{array}{r} \square 8 \square\square \\ \square 5 \square\square \\ +\ \square 8 \square\square \\ \hline \square\square \square\square \end{array}$$

Exercises

1. Give the number for the point above each letter.



2. Find the sums.

[a] $\begin{array}{r} 37 \\ +52 \\ \hline \end{array}$ [b] $\begin{array}{r} 86 \\ +37 \\ \hline \end{array}$ [c] $\begin{array}{r} 534 \\ +256 \\ \hline \end{array}$ [d] $\begin{array}{r} 744 \\ +257 \\ \hline \end{array}$

[e] $57 + 41$ [f] $123 + 574$ [g] $2338 + 5620$

[h] $\begin{array}{r} 217 \\ 314 \\ +253 \\ \hline \end{array}$ [i] $\begin{array}{r} 328 \\ 426 \\ 153 \\ +867 \\ \hline \end{array}$ [j] $\begin{array}{r} 2347 \\ 981 \\ 49 \\ +6 \\ \hline \end{array}$ [k] $\begin{array}{r} 6243 \\ 1527 \\ 4639 \\ +7254 \\ \hline \end{array}$

[l] $\begin{array}{r} 74{,}481 \\ 21{,}225 \\ 96{,}877 \\ 42{,}809 \\ +21{,}688 \\ \hline \end{array}$ [m] $\begin{array}{r} 437{,}639 \\ 990{,}949 \\ 681{,}855 \\ +243{,}286 \\ \hline \end{array}$ [n] $\begin{array}{r} 5337 \\ 6338 \\ 7339 \\ 8330 \\ +9331 \\ \hline \end{array}$ [o] $\begin{array}{r} 27{,}489{,}355 \\ 41{,}315{,}894 \\ 72{,}874{,}056 \\ +98{,}763{,}548 \\ \hline \end{array}$

★ **3.** Rewrite the following addition problems, replacing each ▥ by a digit so that the answers are correct.

[a] $\begin{array}{r} ▥937 \\ 8▥ \\ 109 \\ +4▥85 \\ \hline 85▥7 \end{array}$ [b] $\begin{array}{r} 5{,}74▥ \\ 2▥7 \\ 63 \\ +8{,}▥28 \\ \hline ▥▥{,}174 \end{array}$ [c] $\begin{array}{r} 53{,}73▥ \\ ▥1{,}059 \\ ▥{,}006 \\ +69 \\ \hline ▥31{,}▥▥1 \end{array}$ [d] $\begin{array}{r} 3▥7 \\ 8524 \\ 1▥ \\ +▥35 \\ \hline ▥623 \end{array}$

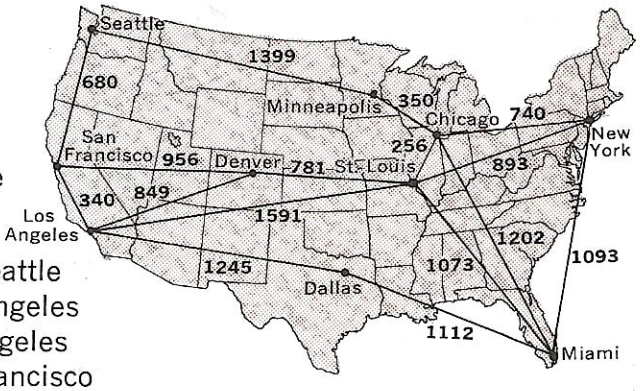
PROBLEMS—AIRLINE DISTANCES

1. Give the total distance for each trip.

[a] Miami-Dallas-Los Angeles
[b] Chicago-St. Louis-Miami
[c] Seattle-Minneapolis-Chicago
[d] Los Angeles-San Francisco-Seattle

2. Find the shortest trip.

[a] New York-Chicago-Minneapolis-Seattle
[b] New York-Chicago-St. Louis-Los Angeles
[c] New York-St. Louis-Denver-Los Angeles
[d] New York-St. Louis-Denver-San Francisco



3. What is the total airline distance covered on a trip from New York to Chicago to Minneapolis to Seattle to San Francisco to Los Angeles to Dallas to Miami to New York?

Think

Study Examples A and B. Try to see a pattern so that you can quickly fill in each ▒ in Exercises 1 and 2. Are there several possibilities for the missing digits?

A

$$\begin{array}{r} 2{,}648 \\ 7{,}351 \\ 3{,}527 \\ +6{,}472 \\ \hline 19{,}998 \end{array}$$

B

$$\begin{array}{r} 5{,}411 \\ 1{,}709 \\ 4{,}588 \\ +8{,}290 \\ \hline 19{,}998 \end{array}$$

1.

$$\begin{array}{r} 6{,}383 \\ \blacksquare{,}\blacksquare\blacksquare\blacksquare \\ 2{,}184 \\ +\ \blacksquare{,}\blacksquare\blacksquare\blacksquare \\ \hline 19{,}998 \end{array}$$

2.

$$\begin{array}{r} 2{,}981 \\ 6{,}553 \\ \blacksquare{,}\blacksquare\blacksquare\blacksquare \\ +\ \blacksquare{,}\blacksquare\blacksquare\blacksquare \\ \hline 19{,}998 \end{array}$$

4-2 FINDING DIFFERENCES

Discussion Exercises

In Examples A and B, some of the digits are covered by screens.

A

$$\begin{array}{r} \blacksquare 7 \blacksquare \\ -\ \blacksquare 4 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$$

B

$$\begin{array}{r} \blacksquare 5 \blacksquare \\ -\ \blacksquare 8 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$$

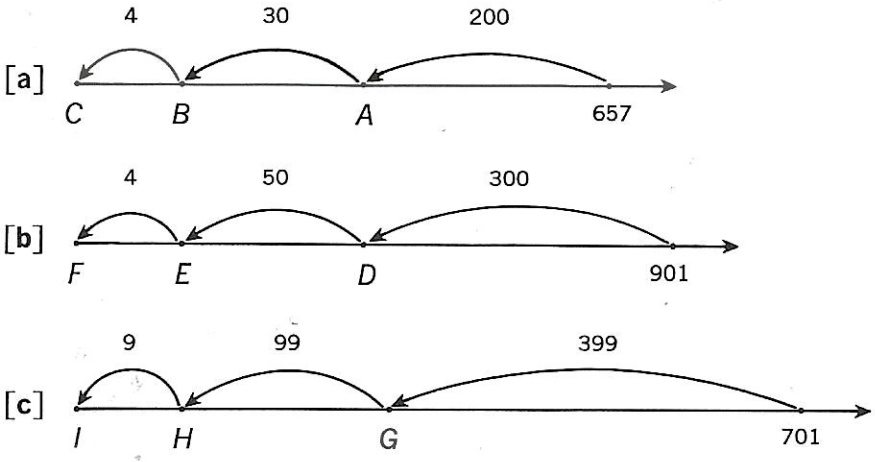
1. Explain why the number of tens in the difference for Example A is either 3 or 2.
2. What are the possible digits for the ▒ in Example B?
3. Give digits for Example A so that the middle digit is 2.
4. Give digits for Example B so that the middle digit is 6.

Exercises

1. Give the digits that are possible for each ▒.

[a] $\begin{array}{r} 6\blacksquare \\ -\ 2\blacksquare \\ \hline 1\blacksquare \end{array}$ [b] $\begin{array}{r} \blacksquare 9 \blacksquare \\ -\ \blacksquare 9 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$ [c] $\begin{array}{r} \blacksquare 69 \\ -\ \blacksquare 7 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$ [d] $\begin{array}{r} \blacksquare 0 \blacksquare \\ -\ \blacksquare 8 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$ [e] $\begin{array}{r} \blacksquare 00 \\ -\ \blacksquare 5 \blacksquare \\ \hline \blacksquare\blacksquare \blacksquare \end{array}$

2. Give the number for the point above each letter.



3. Find the differences.

[a] 87 − 35	[b] 92 − 68	[c] 131 − 49	[d] 862 − 777	[e] 1207 − 849
[f] 5200 − 2637	[g] 7891 − 4987	[h] 9451 − 5789	[i] 23,742 − 19,888	[j] 987,654 − 321,098

[k] 642 − 357 [l] 1028 − 939 [m] 10,000 − 7422
[n] 2344 − 999 [o] 1728 − 1694 [p] 8080 − 7909

4. Solve the equations.

[a] $n - 237 = 629$ [c] $9281 - 7487 = n$ ★[e] $2345 - (1258 - 642) = n$
[b] $683 - n = 409$ ★[d] $(2345 - 1258) - 642 = n$ ★[f] $(2345 - 642) - 1258 = n$

5. Solve the equations.

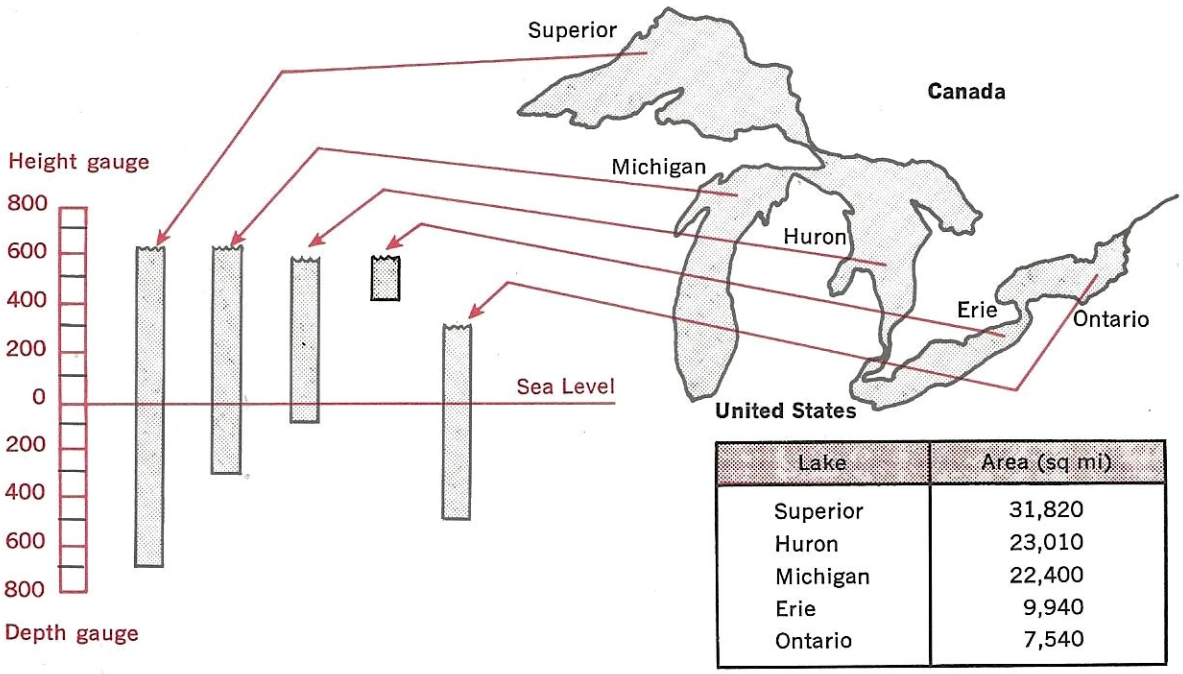
[a] $200 - 99 = n$
[b] $347 - 99 = n$
[c] $1000 - 299 = n$
[d] $5000 - 1999 = n$
[e] $500 - 299 = n$
[f] $9099 - 999 = n$
[g] $8321 - 7999 = n$
[h] $2748 - 1749 = n$

Think

▓ + ▓ = 56
▓ − ▓ = 18

The sum of two numbers is 56. Their difference is 18. What are these two numbers?

PROBLEMS—THE GREAT LAKES



Lake	Area (sq mi)
Superior	31,820
Huron	23,010
Michigan	22,400
Erie	9,940
Ontario	7,540

The five Great Lakes are the largest group of fresh-water lakes in the world. Lake Superior, the largest of the five, is also the largest single fresh-water lake in the world. Only one of the lakes lies entirely within the boundaries of the United States.

The graph shows in feet the height above sea level of the surface of each lake and the depth of each lake as compared to sea level.

1. What is the total area of the five Great Lakes?
2. Estimate to the nearest hundred feet the depth of each of the five lakes.
3. Find the depth below sea level for each lake whose bottom is below sea level.
4. Estimate to the nearest hundred feet the drop in water level from the top of Lake Erie to the top of Lake Ontario.
5. How far above the surface of Lake Ontario is the bottom of Lake Erie?
6. The Caspian Sea, which lies between Europe and Asia in the Caucasus mountains, is the largest inland lake in the world. If the area of the Caspian Sea is 163,500 sq mi, how much larger is it than the total area of the Great Lakes?
7. The second largest fresh-water lake in the world is Lake Victoria in Africa. It has an area of 26,828 sq mi. How much larger is Lake Superior than Lake Victoria? Lake Victoria than Lake Michigan?

4-3 FINDING PRODUCTS

When we multiply factors with more than one digit, we use the distributive principle of multiplication over addition.

$6 \cdot 37$	$=$	$6 \cdot (30 + 7)$	$=$	$(6 \cdot 30) + (6 \cdot 7)$
Step 1 $\begin{array}{r} 37 \\ \times\ 6 \\ \hline 42 \end{array}$		Step 2 $\begin{array}{r} 37 \\ \times\ 6 \\ \hline 42 \\ 180 \\ \hline \end{array}$		Step 3 $\begin{array}{r} 37 \\ \times\ 6 \\ \hline 42 \\ 180 \\ \hline 222 \end{array}$
$6 \cdot 7 = 42$		$6 \cdot 30 = 180$		$42 + 180 = 222$

Explain the steps in this shortcut:

$$\begin{array}{r} {\scriptstyle 4} \\ 37 \\ \times\ 6 \\ \hline 2 \end{array} \rightarrow \begin{array}{r} {\scriptstyle 4} \\ 37 \\ \times\ 6 \\ \hline 222 \end{array}$$

Note how the distributive law is used in the usual shortcut method of multiplication:

$56 \cdot 23$	$=$	$(50 + 6) \cdot 23$	$= (50 \cdot 23) + (6 \cdot 23)$	
Step 1 $\begin{array}{r} 23 \\ \times\ 56 \\ \hline 138 \end{array}$		Step 2 $\begin{array}{r} 23 \\ \times\ 56 \\ \hline 138 \\ 1150 \\ \hline \end{array}$	Step 3 $\begin{array}{r} 23 \\ \times\ 56 \\ \hline 138 \\ 1150 \\ \hline 1288 \end{array}$	Shortcut $\begin{array}{r} 23 \\ \times\ 56 \\ \hline 138 \\ 115 \\ \hline 1288 \end{array}$
$6 \cdot 23 = 138$		$50 \cdot 23 = 1150$	$138 + 1150 = 1288$	

Discussion Exercises

Tell what two factors give the partial product covered by each set of ▥. Then give this partial product.

1. $\begin{array}{r} 271 \\ \times\ 46 \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array}$

2. $\begin{array}{r} 742 \\ \times\ 854 \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array}$

3. $\begin{array}{r} 522 \\ \times\ 708 \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array}$

4. $\begin{array}{r} 906 \\ \times\ 609 \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \\ \hline \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare \end{array}$

Exercises

1. Write the products.

[a] $(10 + 5) \cdot 10$
[b] $(100 + 10 + 2) \cdot 10$
[c] $(30 + 7) \cdot 10$
[d] $(200 + 40 + 5) \cdot 10$
[e] $(10 + 6) \cdot 10$
[f] $(30 + 7) \cdot 100$
[g] $(100 + 4) \cdot 10$
[h] $(300 + 50 + 7) \cdot 100$
[i] $(1000 + 10 + 1) \cdot 10$
[j] $(1000 + 300 + 50 + 4) \cdot 100$

2. Write the products.

[a] $36 \cdot 10$
[b] $45 \cdot 100$
[c] $246 \cdot 10$
[d] $452 \cdot 100$
[e] $2534 \cdot 10$
[f] $2156 \cdot 100$
[g] $2463 \cdot 1000$
[h] $8888 \cdot 1000$

3. Write the products.

[a] $20 \cdot 30$
[b] $200 \cdot 30$
[c] $200 \cdot 300$
[d] $20 \cdot 3000$
[e] $30 \cdot 40$
[f] $300 \cdot 400$
[g] $30 \cdot 400$
[h] $60 \cdot 50$
[i] $80 \cdot 70$
[j] $60 \cdot 600$
[k] $600 \cdot 700$
[l] $220 \cdot 20$
[m] $220 \cdot 30$
[n] $220 \cdot 200$
[o] $132 \cdot 20$
[p] $132 \cdot 300$
[q] $2134 \cdot 200$
[r] $2132 \cdot 2000$

4. Multiply.

[a] $\begin{array}{r} 32 \\ \times 43 \\ \hline \end{array}$
[b] $\begin{array}{r} 32 \\ \times 54 \\ \hline \end{array}$
[c] $\begin{array}{r} 27 \\ \times 58 \\ \hline \end{array}$
[d] $\begin{array}{r} 352 \\ \times 46 \\ \hline \end{array}$

[e] $\begin{array}{r} 257 \\ \times 84 \\ \hline \end{array}$
[f] $\begin{array}{r} 368 \\ \times 29 \\ \hline \end{array}$
[g] $\begin{array}{r} 423 \\ \times 514 \\ \hline \end{array}$
[h] $\begin{array}{r} 367 \\ \times 425 \\ \hline \end{array}$

[i] $\begin{array}{r} 378 \\ \times 496 \\ \hline \end{array}$
[j] $\begin{array}{r} 3524 \\ \times 86 \\ \hline \end{array}$
[k] $\begin{array}{r} 4763 \\ \times 584 \\ \hline \end{array}$
[l] $\begin{array}{r} 8732 \\ \times 4658 \\ \hline \end{array}$

★**5.** Try to discover the shortcut in the two examples. Use this shortcut to work Parts **a** and **b**.

① $\begin{array}{r} 329 \\ \times 123 \\ \hline 987 \\ 3948 \\ \hline 40,467 \end{array}$

② $\begin{array}{r} 713 \\ \times 255 \\ \hline 3565 \\ 17825 \\ \hline 181,815 \end{array}$

[a] $\begin{array}{r} 2714 \\ \times 153 \\ \hline \end{array}$
[b] $\begin{array}{r} 42,345 \\ \times 426 \\ \hline \end{array}$

4-3 FINDING PRODUCTS

When we multiply factors with more than one digit, we use the distributive principle of multiplication over addition.

$6 \cdot 37 = 6 \cdot (30 + 7) = (6 \cdot 30) + (6 \cdot 7)$		
Step 1 37 × 6 42	Step 2 37 × 6 42 180	Step 3 37 × 6 42 180 222
$6 \cdot 7 = 42$	$6 \cdot 30 = 180$	$42 + 180 = 222$

Explain the steps in this shortcut:

$$\begin{array}{r} {\scriptstyle 4} \\ 37 \\ \times\ 6 \\ \hline 2 \end{array} \longrightarrow \begin{array}{r} {\scriptstyle 4} \\ 37 \\ \times\ 6 \\ \hline 222 \end{array}$$

Note how the distributive law is used in the usual shortcut method of multiplication:

$56 \cdot 23 = (50 + 6) \cdot 23 = (50 \cdot 23) + (6 \cdot 23)$			
Step 1 23 × 56 138	Step 2 23 × 56 138 1150	Step 3 23 × 56 138 1150 1288	Shortcut 23 × 56 138 115 1288
$6 \cdot 23 = 138$	$50 \cdot 23 = 1150$	$138 + 1150 = 1288$	

Discussion Exercises

Tell what two factors give the partial product covered by each set of ▮▮▮. Then give this partial product.

1. 271 × 46 = ▮▮▮▮

2. 742 × 854 = ▮▮▮▮▮▮

3. 522 × 708 = ▮▮▮▮▮▮

4. 906 × 609 = ▮▮▮▮▮▮

Exercises

1. Write the products.

[a] $(10 + 5) \cdot 10$
[b] $(100 + 10 + 2) \cdot 10$
[c] $(30 + 7) \cdot 10$
[d] $(200 + 40 + 5) \cdot 10$
[e] $(10 + 6) \cdot 10$
[f] $(30 + 7) \cdot 100$
[g] $(100 + 4) \cdot 10$
[h] $(300 + 50 + 7) \cdot 100$
[i] $(1000 + 10 + 1) \cdot 10$
[j] $(1000 + 300 + 50 + 4) \cdot 100$

2. Write the products.

[a] $36 \cdot 10$
[b] $45 \cdot 100$
[c] $246 \cdot 10$
[d] $452 \cdot 100$
[e] $2534 \cdot 10$
[f] $2156 \cdot 100$
[g] $2463 \cdot 1000$
[h] $8888 \cdot 1000$

3. Write the products.

[a] $20 \cdot 30$
[b] $200 \cdot 30$
[c] $200 \cdot 300$
[d] $20 \cdot 3000$
[e] $30 \cdot 40$
[f] $300 \cdot 400$
[g] $30 \cdot 400$
[h] $60 \cdot 50$
[i] $80 \cdot 70$
[j] $60 \cdot 600$
[k] $600 \cdot 700$
[l] $220 \cdot 20$
[m] $220 \cdot 30$
[n] $220 \cdot 200$
[o] $132 \cdot 20$
[p] $132 \cdot 300$
[q] $2134 \cdot 200$
[r] $2132 \cdot 2000$

4. Multiply.

[a] $\begin{array}{r} 32 \\ \times 43 \\ \hline \end{array}$
[b] $\begin{array}{r} 32 \\ \times 54 \\ \hline \end{array}$
[c] $\begin{array}{r} 27 \\ \times 58 \\ \hline \end{array}$
[d] $\begin{array}{r} 352 \\ \times 46 \\ \hline \end{array}$

[e] $\begin{array}{r} 257 \\ \times 84 \\ \hline \end{array}$
[f] $\begin{array}{r} 368 \\ \times 29 \\ \hline \end{array}$
[g] $\begin{array}{r} 423 \\ \times 514 \\ \hline \end{array}$
[h] $\begin{array}{r} 367 \\ \times 425 \\ \hline \end{array}$

[i] $\begin{array}{r} 378 \\ \times 496 \\ \hline \end{array}$
[j] $\begin{array}{r} 3524 \\ \times 86 \\ \hline \end{array}$
[k] $\begin{array}{r} 4763 \\ \times 584 \\ \hline \end{array}$
[l] $\begin{array}{r} 8732 \\ \times 4658 \\ \hline \end{array}$

★**5.** Try to discover the shortcut in the two examples. Use this shortcut to work Parts **a** and **b**.

① $\begin{array}{r} 329 \\ \times 123 \\ \hline 987 \\ 3948 \\ \hline 40,467 \end{array}$

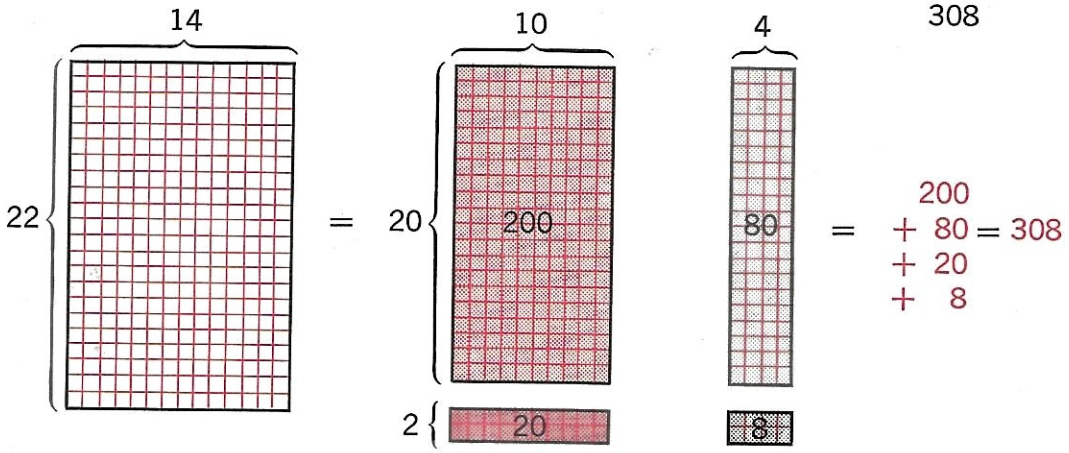
② $\begin{array}{r} 713 \\ \times 255 \\ \hline 3565 \\ 17825 \\ \hline 181,815 \end{array}$

[a] $\begin{array}{r} 2714 \\ \times 153 \\ \hline \end{array}$
[b] $\begin{array}{r} 42,345 \\ \times 426 \\ \hline \end{array}$

6. The diagram that follows gives a geometric meaning to this familiar method of multiplication.

$$\begin{array}{r} 22 \\ \times 14 \\ \hline 88 \\ 22 \\ \hline 308 \end{array}$$

[a] What do the differently colored regions in the diagram correspond to in the multiplication problem?



[b] Make a similar figure for $12 \cdot 13$. What regions correspond to numbers in the usual shortcut?

[c] Make a similar figure for $23 \cdot 15$. To what numbers do the different regions correspond?

7. Give the correct symbol ($>$ or $<$) for each ▒.

[a] $4 \cdot 23$ ▒ 100

[b] $3 \cdot 341$ ▒ 1000

[c] $7 \cdot 308$ ▒ $5 \cdot 400$

[d] $3 \cdot 253$ ▒ 6072

★ **8.** In each equation, is $a = b$, $a > b$, or $a < b$? Neither a nor b stands for zero.

[a] $15 \cdot a = 13 \cdot b$

[b] $a \cdot a = a \cdot b$

[c] $a \cdot b = a \cdot (a + 9)$

[d] $a \cdot b = a \cdot (a - 9)$

★ **9.** Rewrite the following multiplication problems, replacing each ▒ by a digit so that the answers are correct.

[a] ▒▒ × 54 = 3▒34

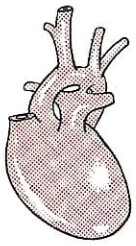
[b] 99 × ▒▒ = 22▒▒

[c] ▒▒▒ × 77 = 27,6▒▒

[d] ▒▒▒ × 999 = 47▒,▒▒4

PROBLEMS—THE HUMAN BODY

1. Your heart, under normal conditions, beats about 72 times every minute. About how many times does your heart beat during
[a] 1 hr?
[b] a 24-hr day?
[c] a 365-day year?
[d] a 70-year lifetime?



2. We respire (inhale and exhale) at a rate of about 18 times per minute. How many times does our heart beat during one respiration period? (Use heartbeat rate of Exercise 1.)

3. An adult man weighs 160 lb. An adult whale can weigh 1250 times as much.
[a] How many pounds can an adult whale weigh?
[b] About how many tons is this?

4. A person produces about 65 calories of heat per hour while sleeping. How many calories are produced by a person who sleeps from 9:30 P.M. to 7:30 A.M.?

5. A student produces about 170 calories of heat per hour while sitting at a desk in the classroom. How many calories are produced during a 6-hr school day?

6. The brain of a baby weighs about 12 ounces (oz). The weight of a baby is 12 times his brain weight. About how many pounds does the baby weigh?

7. An adult weighs about 35 times his brain weight. If a man's brain weight is 5 lb, what is his body weight?

8. The body of a 160-lb man consists of about 105 lb of oxygen, 29 lb of carbon, 16 lb of hydrogen, and the remainder consists of various other elements. How many pounds of other elements does his body contain?

★**9.** The blood in the human body contains about 35,000,000,000,000 blood cells. Ten million of these cells are made and destroyed every second.
[a] How many cells must the body manufacture every 24 hr in order to keep the number of blood cells the same?
[b] About how many days must elapse before all the blood cells in the body are replaced at least once?

★**10.** Signals travel in our nervous system at the rate of 300 ft per sec. How far would a nervous system signal travel in an hour?

★**11.** Sound travels about 742 mph at sea level.
[a] Do signals in our nervous system travel faster or slower than sound in air?
[b] How much faster or slower?

Think

A man weighs twice as much as his wife. His wife weighs twice as much as their son. Their total weight is 399 lb. How much does each person weigh?

4-4 EXPONENTS

So far, you have used exponents mainly with powers of ten. You can use exponents in the same way with other numbers.
Study the examples.

Symbol	We think	We say
8^2	$8 \cdot 8$ Two factors of 8	8 squared or 8 to the second power
5^4	$5 \cdot 5 \cdot 5 \cdot 5$ Four factors of 5	5 to the fourth power
7^3	$7 \cdot 7 \cdot 7$ Three factors of 7	7 cubed or 7 to the third power
11^5	$11 \cdot 11 \cdot 11 \cdot 11 \cdot 11$ Five factors of 11	11 to the fifth power

These examples show how we use exponents.

$a^1 = a$ $a^2 = a \cdot a$ $a^3 = a \cdot a \cdot a$	$a^4 = a \cdot a \cdot a \cdot a$ $\vdots$ $a^n = \underbrace{a \cdot a \cdot a \ldots}_{n \text{ factors of } a}$

Exercises

1. Write each product, using exponents.

Example: $7 \cdot 7 \cdot 7 = 7^3$

[a] $4 \cdot 4$ [c] $10 \cdot 10 \cdot 10$ [e] $2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
[b] $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$ [d] $15 \cdot 15 \cdot 15 \cdot 15 \cdot 15 \cdot 15$ [f] $5 \cdot 4 \cdot 5 \cdot 4 \cdot 5 \cdot 4 \cdot 5$

2. Write each number in expanded form and find the result.

Example: $8^5 = 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 32{,}768$

[a] 16^2 [c] 2^8 [e] 3^4 [g] 13^3 ★[i] $4^0 \cdot 3^0 \cdot 2^0$
[b] 4^4 [d] 9^2 [f] 17^1 [h] $2^3 \cdot 3^2 \cdot 4^1$ ★[j] $4^3 \cdot 3^2 \cdot 2^1 \cdot 1^0$

3. Use the words **base** and **exponent** to complete the sentence correctly. When we write 6^3, we call 6 the _?_ and 3 the _?_, and we write: $6^3 = 6 \cdot 6 \cdot 6 = 216$.

4. Solve the equations.

[a] $5^3 = n$ [b] $3^3 = h$ [c] $5^5 = y$ [d] $2^6 = a$ [e] $13^2 = r$ [f] $4^5 = c$ [g] $9^3 = t$ [h] $8^4 = b$ [i] $4^8 = s$ [j] $1^{15} = m$ [k] $23^1 = w$ ★[l] $4^3 = 2^n$

5. Find the products.

[a] $2^3 \cdot 3^2$ [b] $5^2 \cdot 10^2$ [c] $7 \cdot 3^3$ [d] $2^4 \cdot 2^3$ [e] $10^3 \cdot 10^2$ [f] $2^2 \cdot 3^2 \cdot 5^2$ [g] $7^1 \cdot 11^1 \cdot 13^1$ [h] $8^2 \cdot 11^2$

6. Give the correct symbol ($>$ or $<$) for each ◍.

[a] 2^3 ◍ 3^2 [b] 5^4 ◍ 8^3 [c] 9^3 ◍ 5^4 [d] 2^{10} ◍ 10^3

Think

Find two different numbers a and b so that $a^b = b^a$.

★**7.** What number is 2^{3^2}?

4-5 MENTAL COMPUTING

While man may compute slowly, he still is the only computer capable of analyzing and computing without external direction. Thus, skill in computing is an important part of solving problems.

Exercises

1. Find the largest whole number for n that will make the sentence true. Part a is completed for you.

[a] $n \cdot 5 < 18$ (Answer: $4 \cdot 5 = 20$; $3 \cdot 5 = 15$. We choose $n = 3$.)

[b] $n \cdot 4 < 30$ [c] $n \cdot 9 < 71$ [d] $n \cdot 8 < 52$ [e] $n \cdot 7 < 66$ [f] $n \cdot 6 < 16$ [g] $n \cdot 7 < 39$ ★[h] $n \cdot 9 < 1$ ★[i] $n \cdot n < n + n$

2. From the set of numbers {10, 20, 30, 40, 50, 60, 70, 80, 90}, select the largest number that will make the sentence true.

[a] $n \cdot 6 < 307$ [b] $n \cdot 8 < 658$ [c] $n \cdot 9 < 555$ [d] $n \cdot 5 < 127$ [e] $n \cdot 4 < 281$ [f] $n \cdot 3 < 179$ [g] $n \cdot 7 < 644$ [h] $n \cdot 6 < 444$

3. Find the largest whole number for n that will make the sentence true.

[a] $n \cdot 32 < 137$
[b] $n \cdot 57 < 381$
[c] $n \cdot 23 < 118$
[d] $n \cdot 83 < 681$
[e] $n \cdot 91 < 508$
[f] $n \cdot 48 < 380$
[g] $n \cdot 89 < 740$
[h] $n \cdot 37 < 351$
[i] $n \cdot 66 < 217$
[j] $n \cdot 72 < 584$
★[k] $n \cdot 714 < 2347$
★[l] $n \cdot 579 < 3827$

★**4.** Give the correct symbol (>, <, or =) for each ●.

[a] $6 \cdot 27 \bullet 161$
[b] $9 \cdot 39 \bullet 353$
[c] $5 \cdot 72 \bullet 354$
[d] $8 \cdot 47 \bullet 376$
[e] $3 \cdot 45 \bullet 131$
[f] $4 \cdot 99 \bullet 396$
[g] $6 \cdot 66 \bullet 408$
[h] $7 \cdot 31 \bullet 218$
[i] $9 \cdot 56 \bullet 500$

5. Rewrite each multiplication problem, replacing each ▮ by a digit so that the answers are correct.

[a] $2▮ \times 4 = 92$
[b] $▮▮ \times 6 = 78$
[c] $23 \times ▮▮ = 368$
[d] $42 \times 3▮ = 2▮0 + 126 = 1▮▮0$
[e] $36 \times 1▮ = 1▮0 + 36 = ▮▮0$

[f] $37 \times 2▮ = ▮4▮ + 74 = 88▮$
★[g] $▮▮ \times 43 = ▮▮ + ▮▮▮ = 11▮8$
★[h] $▮▮ \times 42 = 7▮ + ▮▮▮ = ▮▮▮0$
★[i] $▮▮ \times 6▮ = ▮▮1 + ▮▮2 = 29▮1$
★[j] $▮▮ \times ▮3 = 22▮ + 3▮▮ = 3▮7▮$

6. Study the examples. Then compute the products mentally.

(A) $101 \cdot 73$
$101 \cdot 73 = (100 + 1) \cdot 73$
$= 7300 + 73$
$= 7373$

(B) $99 \cdot 73$
$99 \cdot 73 = (100 - 1) \cdot 73$
$= 7300 - 73$
$= 7227$

[a] $101 \cdot 43$
[b] $99 \cdot 43$
[c] $101 \cdot 87$
[d] $99 \cdot 87$
[e] $102 \cdot 34$
[f] $98 \cdot 34$
[g] $1001 \cdot 65$
[h] $999 \cdot 65$

★ **7.** Compute the products mentally.

[a] 102 · 23	[d] 999 · 357	[g] 998 · 423	[j] 1002 · 506
[b] 98 · 23	[e] 1001 · 886	[h] 1002 · 423	[k] 998 · 506
[c] 1001 · 357	[f] 999 · 886	[i] 99 · 317	[l] 997 · 506

4-6 DIVIDING

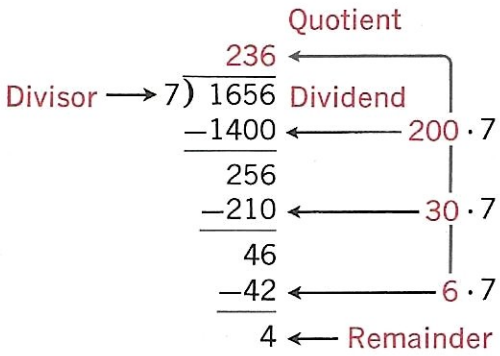
Discussion Exercises

1. In this example, we have subtracted a total of 236 sevens.

[a] How many sevens were subtracted the first time?

[b] How many sevens were subtracted the second time?

[c] How does the remainder compare in size to the divisor?



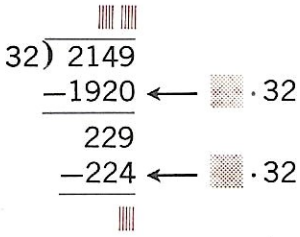
2. In this example, some of the digits are covered by screens.

[a] How many 32's were subtracted the first time?

[b] How many 32's were subtracted the second time?

[c] What is the quotient?

[d] What is the remainder?



3. When the divisor is a number less than 10, we often use a method called "short division" to find the quotient. Study Examples A and B.

A

$$4\overline{)3309}\quad 8$$

We think about hundreds:
33 ÷ 4,
quotient 8,
remainder 1.

$$4\overline{)3309}\quad 82$$

We think about tens:
10 ÷ 4,
quotient 2,
remainder 2.

$$4\overline{)3309}\quad 827,\ \text{R1}$$

We think about units:
29 ÷ 4,
quotient 7,
remainder 1.

B Here is the way a student's work might appear. Explain how the student found each digit in the quotient.

$$7\overline{)9^{2}3^{2}8\ 5^{5}2}\quad 1\,3,4\,0\,7\text{R}3$$

Exercises

1. Each exercise has been partially worked. Copy and complete the work to find the quotient.

[a] $8\overline{)\,3846}$ (400)
-3200
646

[b] $42\overline{)\,7489}$ ([redacted])
-4200
3289

[c] $59\overline{)\,13{,}806}$ ([redacted])
$-11{,}800$

[d] $212\overline{)\,5\,3{,}6\,4\,8}$ (200)
$-$[redacted]

2. Find the quotients and remainders, using short division.

[a] $7\overline{)\,246}$
[b] $9\overline{)\,314}$
[c] $6\overline{)\,4455}$
[d] $5\overline{)\,3646}$
[e] $8\overline{)\,2148}$
[f] $7\overline{)\,18{,}488}$
[g] $9\overline{)\,61{,}407}$
[h] $7\overline{)\,1001}$
[i] $3\overline{)\,12{,}345}$
[j] $4\overline{)\,81{,}624}$
[k] $9\overline{)\,2300}$
[l] $4\overline{)\,70{,}771}$
[m] $5\overline{)\,36{,}628}$
[n] $2\overline{)\,3{,}584{,}286}$
[o] $9\overline{)\,111{,}111{,}111}$
[p] $3\overline{)\,186{,}001}$

★**3.** If the remainder is zero, what is the units digit of each dividend? What is the quotient?

[a] $7\overline{)\,482{,}35$[redacted]
[b] $8\overline{)\,34{,}15$[redacted]
[c] $6\overline{)\,31{,}22$[redacted]

★**4.** If the remainder is zero, there are two possibilities for the units digit in each dividend. Give both possibilities.

[a] $6\overline{)\,31{,}51$[redacted]
[b] $4\overline{)\,71{,}43$[redacted]
[c] $6\overline{)\,3{,}00$[redacted]

★**5.** What division fact is shown by the repeated subtraction?

? ÷ ?

5313
-4600
713
-690
23
-23
0

Think

What are the digits in the division problem? Each letter stands for a single digit.

$$\begin{array}{r} bb \\ a\overline{)\,bcc} \\ a \\ \hline bc \\ a \\ \hline b \end{array}$$

4-7 AVERAGE (ARITHMETIC MEAN)

Bob is a student at Kennedy Junior High School. Bob's teachers and their ages are shown in the table.

Bob found the sum of the teachers' ages. Then he noticed that if each of his six teachers were 39 years old, the sum of their ages would be the same.

Name	Subject taught	Age
Mr. DeWitt	Physical Education	26
Mrs. Mays	English	29
Mr. Harding	Mathematics	34
Mr. Kurdy	Social Studies	59
Mr. Reed	Science	48
Mr. Stakowsky	Music	38

$$\begin{array}{ccccccccccccc} 26 & + & 29 & + & 34 & + & 59 & + & 48 & + & 38 & = & 234 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ 39 & + & 39 & + & 39 & + & 39 & + & 39 & + & 39 & = & 6 \cdot 39 = 234 \end{array}$$

We say that the **average**, or the **arithmetic mean**, of the ages is 39. How would you find the average age of Bob's teachers?

Exercises

1. Bob's scores on four mathematics tests were 83, 78, 92, and 87.

[a] What is the sum of his scores?

[b] What is the arithmetic mean of his scores?

2. After its first five games, the Kennedy Junior High School basketball team had the record shown on the scoreboard.

Scoreboard	
Kennedy	Opponents
48	46
54	39
52	53
60	50
56	61

[a] What is the average number of points scored by the Kennedy team?

[b] To the nearest whole number, what is the average number of points scored by the opponents?

[c] What is the average number of points by which the Kennedy team has outscored their opponents in the three games they have won?

[d] What is the average number of points by which the Kennedy team has been defeated in the two games they have lost?

3. Joe Tallman is the team's leading scorer. In the first five games, he scored 16, 21, 13, 17, and 12 points. To the nearest whole number, what is Joe's average number of points for each game?

4. Joan found that six of her friends lived farther away from school than she did. Joan then made the following record.

Name	Miles distant
Jean	4
Stan	3
Harry	7
Alice	5
Charley	9
Anne	8

What is the average number of miles away from school that Joan's six friends lived?

5. For five weeks, Bill kept track of the number of pages he read in all classes. He then made the following tally.

Week	Pages
First	23
Second	47
Third	32
Fourth	59
Fifth	38

To the nearest whole number, what is the average number of pages which Bill read each week?

6. Four of Don's teammates weighed 130, 144, 138, and 152 lb respectively. What is the average weight of the four boys?

Think

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

1. What is the average of the calendar dates shown along each arrow?

2. Try this problem for another three-by-three square on the calendar.

4-8 SHORTCUTS IN DIVIDING

We can find the quotient for $67\overline{)15{,}949}$ by finding one-digit quotients for

$67\overline{)159}$, $67\overline{)254}$, and $67\overline{)539}$

Study the steps below.

Step 1	Step 2	Step 3
$\begin{array}{r} \quad 2 \\ 67\overline{)15{,}949} \\ -134 \\ \hline 25 \end{array}$	$\begin{array}{r} \quad 23 \\ 67\overline{)15{,}949} \\ -134 \\ \hline 254 \\ -201 \\ \hline 53 \end{array}$	$\begin{array}{r} \quad 238 \\ 67\overline{)15{,}949} \\ -134 \\ \hline 254 \\ -201 \\ \hline 539 \\ -536 \\ \hline 3 \end{array}$
Dividing hundreds	Dividing tens	Dividing ones

We can find the quotient for any division problem by finding the one-digit quotients for a series of simple division problems.

Discussion Exercises

1. Tell how you find the single-digit quotients in Steps 1, 2, and 3 of the preceding table.

2. [a] Tell what three "simple" division problems were worked in finding $83\overline{)26{,}916}$.
[b] Which of the three problems involved "dividing tens"?
[c] Which problem involved "dividing ones"?

$$\begin{array}{r} \quad 324 \\ 83\overline{)26{,}916} \\ -249 \\ \hline 201 \\ -166 \\ \hline 356 \\ -332 \\ \hline 24 \end{array}$$

3. Complete the division problem and explain each step.

$29\overline{)747}$ → Think: $30\overline{)747}$, $\begin{array}{r} 2 \\ 60 \end{array}$ → Write: $\begin{array}{r} \quad 2 \\ 29\overline{)747} \\ 58 \end{array}$

4. Complete the following division problems and explain each step.
[a] $88\overline{)2392}$ [b] $42\overline{)1216}$ [c] $76\overline{)3027}$

Exercises

1. Estimate the first digit in the quotient by first rounding the divisor to the nearest ten.

[a] $19\overline{)564}$ [d] $68{,}235 \div 47$ [g] $64\overline{)2139}$ ★[j] $36\overline{)3024}$

[b] $52\overline{)7922}$ [e] $1147 \div 19$ [h] $77\overline{)61{,}256}$ ★[k] $75\overline{)5101}$

[c] $86\overline{)2395}$ [f] $49{,}221 \div 53$ [i] $91\overline{)8333}$ ★[l] $43\overline{)1307}$

2. Estimate the quotients.

[a] $32\overline{)96}$ [d] $77\overline{)616}$ [g] $374 \div 48$ ★[j] $84\overline{)2216}$

[b] $47\overline{)97}$ [e] $434 \div 62$ [h] $67\overline{)629}$ ★[k] $17\overline{)698}$

[c] $56\overline{)112}$ [f] $83\overline{)664}$ [i] $99\overline{)505}$ ★[l] $42\overline{)1029}$

3. Estimate the tens digit in each quotient.

Example: $48\overline{)2762}$

Think: Round 48 to 50; then 276 ÷ 50 gives the tens-digit 5.

[a] $11\overline{)143}$ [d] $625 \div 25$ [g] $37\overline{)1234}$ ★[j] $72\overline{)5143}$

[b] $15\overline{)195}$ [e] $91\overline{)4186}$ [h] $4576 \div 87$ ★[k] $89\overline{)2841}$

[c] $37\overline{)444}$ [f] $1257 \div 43$ [i] $35\overline{)963}$ ★[l] $37\overline{)811}$

4. Estimate the quotients.

Example: $600 \div 99$

Think: $600 \div 100 = 6$

6 is the required answer.

[a] $300 \div 99$ [d] $800 \div 199$ [g] $4000 \div 998$ [j] $4000 \div 198$

[b] $8000 \div 999$ [e] $7999 \div 99$ [h] $2400 \div 98$ [k] $6101 \div 102$

[c] $3600 \div 99$ [f] $9001 \div 99$ [i] $9000 \div 101$ [l] $4312 \div 99$

5. Find the quotients and remainders.

[a] $61\overline{)2551}$ [b] $88\overline{)6050}$ [c] $36\overline{)6158}$

[d] $43\overline{)1776}$ [e] $46\overline{)1776}$ [f] $75\overline{)4338}$

[g] $94\overline{)12{,}716}$ [h] $18\overline{)44{,}319}$ [i] $37\overline{)21{,}515}$

[j] $512\overline{)23{,}849}$ [k] $681\overline{)46{,}464}$ [l] $902\overline{)38{,}574}$

Think: $500\overline{)23{,}849}$ Think: $700\overline{)46{,}464}$ Think: $900\overline{)38{,}574}$

[m] $777\overline{)111{,}111}$ [n] $832\overline{)190{,}851}$ [o] $429\overline{)312{,}498}$

6. Replace each ▥ by a digit so that the answers are correct. All remainders are zero.

[a] $12\overline{)▥▥▥}$ → 34; ▥ 3 ▥; 4 ▥; 4 8

[b] $3▥\overline{)▥2▥}$ → 2▥; ▥ ▥; 16▥; 16▥

★[c] $84\overline{)▥▥▥}$ → ▥▥; ▥ ▥; ▥4

★[d] $▥▥\overline{)3▥▥8}$ → ▥3▥; ▥ ▥; 8▥; ▥2; 1▥8; 1▥8

SHORT STORIES

1. 396 students. 9 school buses. Same number of students in each bus. How many students in each bus?

2. Strawberries at $10.80 a crate. 24 boxes in a crate. How much for one box?

3. Trip: 576 mi. Time: 9 hr. What was the average speed in miles per hour?

4. Three boys on a scale. Total weight: 231 lb. What is the average weight?

5. Each lawn mowed for $1.75. Earnings: $14. How many lawns were mowed?

6. Satellite in orbit goes 18,000 mph. How far in 15 hr?

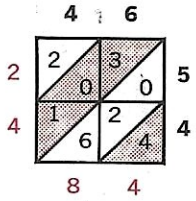
7. A car gets 17 mi per gallon of gasoline. Travels 646 mi. How many gallons of gas used?

8. 220-yd dash. Time: 26 sec. About how many **feet** per second?

9. Car goes 105 mi at 35 mph, then 385 mi at 55 mph. How many hours in all?

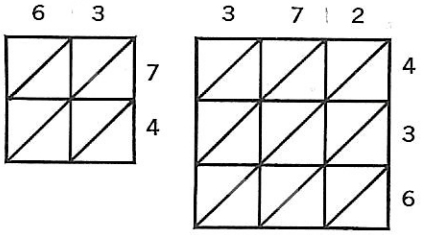
10. Dodgers played 73 baseball games. Scored 366 runs. About how many runs per game?

Think



The diagram shows an old method of multiplying **46** and **54** to get the product 2484.

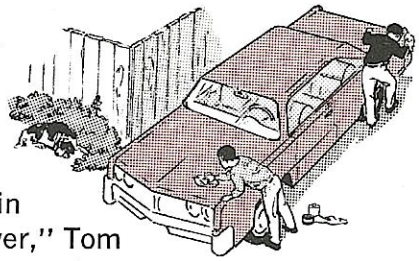
Using this method, find these products. →



4-9 REASONING IN PROBLEM SOLVING

Students in a seventh-grade class were studying word problems. Their teacher gave them the following problem.

> Jim can wax a car in 6 hours. Bill can wax a car in 3 hours. If they work together how long will it take them to wax one car?



Ted answered. "That's easy; the answer is 9 hours." Some of the students in the class nodded their heads in agreement, but Tom disagreed. "I don't know the answer," Tom said, "but I can prove that 9 hours is wrong." What argument do you think Tom gave to convince the class?

Discussion Exercises

For each problem, a **wrong answer** is given. Explain why the given answer is wrong and then estimate the correct answer.

1. Harry had 20 hamsters. All but 7 died. How many were left? (**Answer**: 13)
2. How many inches are there in 48 ft? (**Answer**: 4)
3. A baseball team has played 150 games. It has won 30 more games than it has lost. How many games has it lost? (**Answer**: 120)

4. A ring cost \$60. A watch cost \$40 more than the ring. What was the total cost of the watch and ring? **(Answer:** \$100)

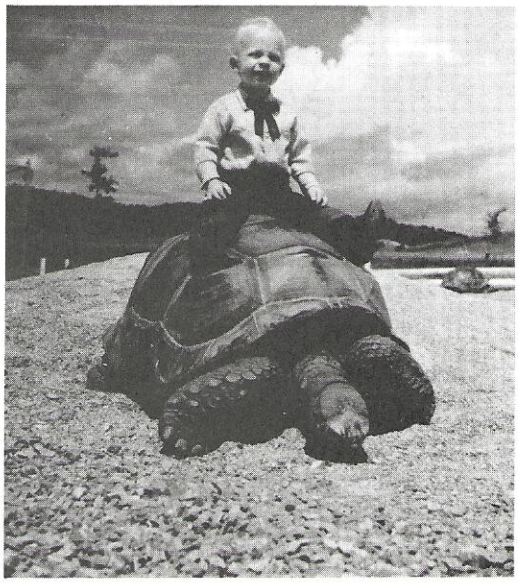
5. A watch and a ring cost \$140. The watch cost \$40 more than the ring. How much did each cost? **(Answer:** watch \$100, ring \$40)

6. Joan is now 6 years old, and Ed is twice as old as Joan. When Joan is 12, how old will Ed be? **(Answer:** 24)

7. Joe had \$20. He spent \$2.00 in one store and half of the remainder of his money in a second store. How much money did he have left? **(Answer:** \$8.00)

8. An archery set and a tennis racket together cost \$48. The archery set cost three times as much as the tennis racket. How much did the tennis racket cost? **(Answer:** \$16)

9. The average life of a man is about 2,000,000,000 sec. The giant turtle lives $\frac{1}{2}$ again as long as man. How many seconds is the average life of the giant turtle? **(Answer:** 1,000,000,000)



10. It is 74 mi from town *A* to town *B*. It is 32 mi from town *B* to town *C*, and town *C* is on the road between *A* and *B*. How far is it from *A* to *C*? **(Answer:** 106 mi)

11. One pipe can fill a tank in 15 minutes (min). A second pipe can fill the tank in 12 min. How long will it take to fill the tank if both pipes are used? **(Answer:** 27 min)

12. Doug weighs 100 lb while standing on one foot on a scale. How much will he weigh if he stands on both feet? **(Answer:** 200 lb)

13. Six years ago Sam was 6 and Joe was 18. What is the number by which you must multiply Sam's age to get Joe's age now? **(Answer:** 3)

★**14.** A tank can be filled in 15 min by pumping water through one pipe. A drain pipe makes it possible to empty the tank in 10 min. The tank is full of water and the drain pipe is opened at the same moment at which the pump for filling the tank is turned on. How long will it take the tank to empty? **(Answer:** 5 min)

4-10 WORDLESS PROBLEMS

Often a sketch or a diagram will help you understand a problem that might otherwise be difficult. Each sketch or diagram below should suggest a problem. For each picture, write and solve your own problem.

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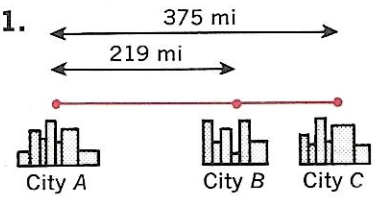

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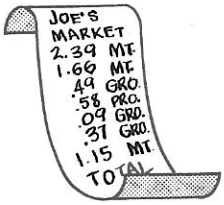

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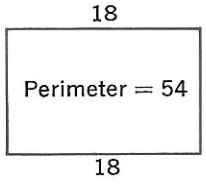

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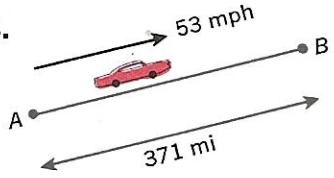

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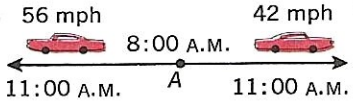

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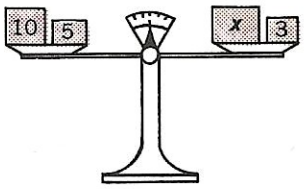
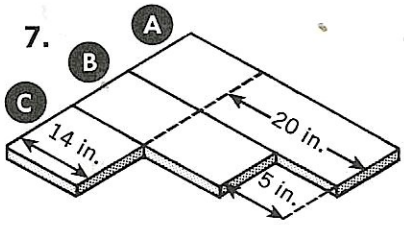


Balance scale

7.



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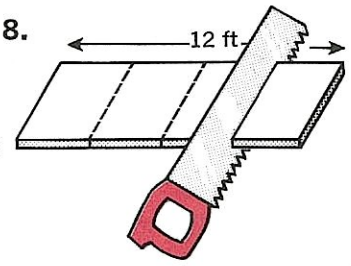

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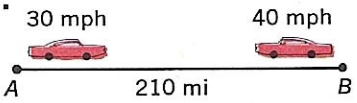

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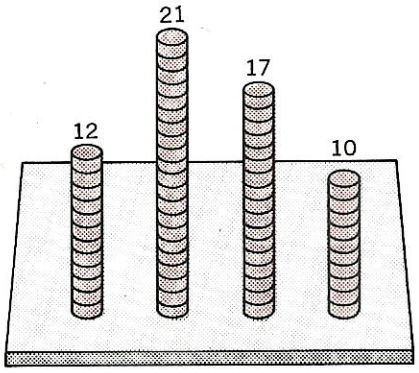


Think

A man fenced his garden so that the fence had the form of a square. When he finished, there were 10 fence posts on each side. How many posts did he use?

4-11 DRAWING DIAGRAMS IN PROBLEM SOLVING

Diagrams will often help you solve problems. The quality of your art work is not important. The drawings help you form a clear mental picture of the important parts of the problem.

The exercises illustrate the use of diagrams in solving problems.

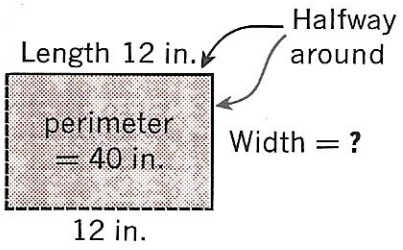
Discussion Exercises

1. The perimeter (distance around) of a rectangle is 40 in. The rectangle is 12 in. long. How wide is the rectangle? The questions may help you find the answer.

[a] The sketch shows the things that are known about the rectangle. What is the distance **halfway** around the rectangle?

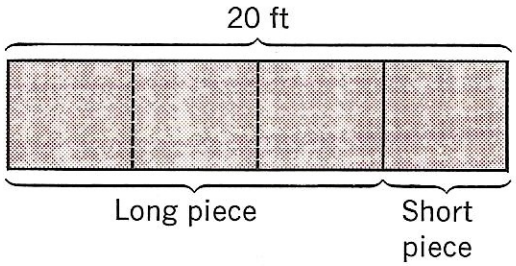
[b] If half the perimeter is 20 in., what is the width of the rectangle?

[c] Can you think of a different way of finding the width?



2. A board 20 ft long is cut into two pieces. One piece is three times as long as the other. How long is each piece? Consider the following discussion.

The diagram shows the things that we know about the problem. The two dotted lines through the long piece help us to remember that the long piece is three times as long as the short piece. How many pieces, the length of the short piece, are contained in the 20-ft board?



Exercises

Draw a diagram that will help you analyze the problem. Then solve the problem.

1. A board is cut into two pieces. The shorter piece is 8 in. long, and the longer piece is twice as long. How long was the board before cutting?

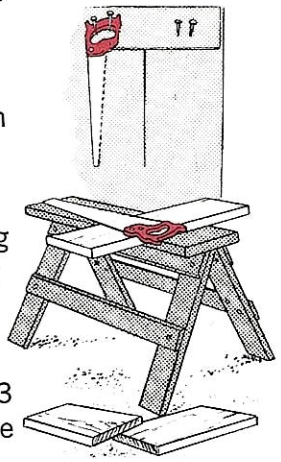
2. The perimeter of a rectangle is 50 ft and its width is 8 ft. How long is the rectangle?

3. The length of a rectangle is 4 in. longer than its width, and the perimeter of the rectangle is 56 in. Find the length and width of the rectangle.

4. A 36-in. board is cut into two pieces. One piece is twice as long as the other. How long is each piece?

5. A 14-ft board is cut into three pieces. Two of the pieces have the same length and the third is 2 ft longer than each of the others. How long is each piece?

6. A triangle has two sides of equal length. The perimeter of the triangle is 20 in., and the length of one of the two sides of equal length is 8 in. How long is the third side?



7. The perimeter of a triangle having two sides of the same length is 42 in. The third side is 6 in. longer than each of the others. How long is each side?

8. A board is cut into three pieces. The shortest piece is half as long as the next longer piece. The longest piece is 3 ft longer than the middle piece. The board was originally 18 ft long. How long is each piece?

9. There are two stacks of checkers on a table. If you moved 3 checkers from the taller pile to the shorter one, there would be the same number of checkers in each pile. How many more are there in the tall pile than in the short one?

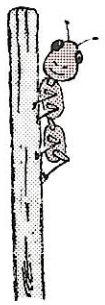
10. There are two rows of cards on a table. One row contains 12 cards. If we moved 5 cards from this row to the second row, the second row would have twice as many cards as the first. How many cards are in the second row?

11. Two cars are 100 mi apart.

[a] If they drive toward each other and meet in 2 hr, 40 mi from the starting point of the slower car, how fast is each car traveling?

★[b] If they drive in the same direction and in 2 hr the faster car passes the slower one 40 mi from the starting point of the slower car, what is the speed of each car?

Think



A bug tries to climb an 8-ft pole. Each day he climbs up 2 ft. Each night while he sleeps he slips down 1 ft. When will he reach the top of the pole?

4-12 RECOGNIZING NEEDED INFORMATION

When you read the statement of a problem, you must recognize what information is important. Sometimes the statement may omit facts that are needed to work the problem. For example, you may need to know the number of feet in one rod, and this information may not be given in the problem itself. Sometimes extra information **not needed** to solve the problem is included, and you must sort out the important ideas from the unimportant ones.

Exercises

1. None of these problems contains enough information to work the problem. What extra facts are needed?

[a] Strawberries sell for 45¢ a quart. What does a crate of strawberries cost?

[b] There are 90 students in the seventh grade at Center Junior High. They plan to use buses on a field trip. How many will they need?

[c] It takes 4 boys and 4 girls to form one set for square dancing. How many sets can be formed from 64 students?

[d] Sue and Jane together have $2.95. Jane spent 98¢ for a pair of socks. How much money does Jane have left?

[e] A farmer had 500 bushels (bu) of wheat. He sold a wagon load containing 6900 lb of wheat. How many bushels had he left?

[f] Students in the seventh-grade class sold 180 tickets for the class play. Adult tickets were $1.00 and student tickets 50¢. How much did they earn for the class treasury?

2. In each of these problems, information that is not needed is given. Pick out unnecessary information.

[a] A car is traveling at 50 mph. It consumes gasoline at the rate of 25 mi per gallon. How long will it take to travel 150 mi?

[b] Terry sells newspapers for 5¢ each and gets 1¢ of the sales price for his commission. He is saving what he earns to buy a bike which costs $48. If on the average he earns $12 per week, how many papers must he sell in order to pay for the bicycle?

[c] A number is multiplied by 5 and to this product 4 is added. The original number is an odd number less than 10 and the final result is 39. Find the original number.

[d] Mary has three United States coins in her pocket, which together have a value of 31¢. One of them is a quarter. What are the coins?

[e] A set of five United States coins has a value of 51¢. There is only one penny in the set. What are the other coins?

4-13 USING COMMON SENSE—ESTIMATING

In most problems that we work, we can come very close to the correct answers by thinking carefully and making estimates.

A If we are told that a 45-acre field of corn produces 99 bushels per acre, we can reason:

99 is almost 100; therefore the field produces about $45 \cdot 100$, or 4500 bushels.

B If ten numbers are to be added and every number is between 60 and 100, we can reason:

If each number were 80, the sum would be about the same. Hence, it is about $10 \cdot 80$, or 800.

Exercises

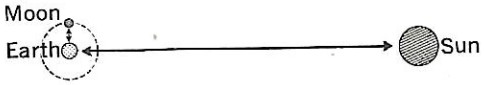
1. After each problem, we have listed three possible answers. None is correct, but just one of the three will be quite close to the correct result. Use estimation to pick out this answer.

[a] A farmer raised 87 bu of corn per acre on 102 acres. What was his total crop? (**Answers:** 900 bu; 90,000 bu; 9000 bu)

[b] A baseball team played 182 games. It won 24 more than it lost. How many games did it win? (**Answers:** 206; 100; 80)

[c] For every 4 students at a track meet, there were 3 adults. If 124 student tickets were sold, how many adult tickets were sold? (**Answers:** 40; 90; 150)

[d] The distance from the earth to the sun is about 93,000,000 mi, and that from the earth to the moon is about 250,000 mi. About how many times as far is it from the earth to the sun as from the earth to the moon? (**Answers:** 4,000,000; 4000; 400)



[e] What is the sum of the whole numbers from 1 to 100? (**Answers:** 500; 5000; 50,000)

[f] The letter "e" occurs more frequently in writing than any other letter in the alphabet. Chris counted the number of times "e" occurred in each of 10 lines of type. Here is the data.

Line	1	2	3	4	5	6	7	8	9	10
Number of e's	5	8	6	8	2	6	5	7	5	7

About how many times, on the average, did "e" occur in each line? (**Answers:** 4; 6; 8)

★[g] Suppose that a rocket ship can travel from the earth to Mars in 260 days. It takes 45 years and 273 days to reach the planet Pluto. Approximately how many round trips could one rocket ship make between the earth and Mars while a second ship was traveling to Pluto? (**Answers:** 30; 80; 6000)

2. Estimate answers to the following problems.

[a] What is the product of 987 and 864?

[b] How much are 315 bu of soybeans worth if the market price is $2.89 a bushel?

[c] A cubic foot of water weighs about 62 lb. A cubic foot of gold weighs about 19 times as much. About how much does a cubic foot of gold weigh?

[d] A jet airplane travels at 600 mph and flies nonstop from Chicago to Miami, Florida. If the airline distance is 1202 mi, about how long will the flight take?

★ [e] A satellite is traveling about 6 mi per sec. About how far will it travel in an hour?

★ [f] A team played 162 baseball games in a season and lost 20 more than it won. About how many games did it win?

★ [g] A factory worker earns $132.50 per week. His weekly deductions for taxes, insurance, and so forth, amount to $13.67. What do you estimate is his take-home pay for a year?

★ [h] The conservation department in a state caught 1000 fish in a small lake, marked these fish, and dumped them back into the water. A few months later they caught 1000 more fish from this lake. They found that 22 of these 1000 fish were marked. Estimate the total number of fish in the lake.

3. Think of each exercise as the beginning of a word problem. Using these beginnings, make up word problems of your own. Solve each of your word problems.

[a] A jet transport can travel 610 mph. It is leaving the airport at Los Angeles to fly to New York. It is 2400 mi from New York to Los Angeles.

[b] There are 93 seventh graders, 81 eighth graders, 80 ninth graders, 75 tenth graders, 68 juniors, and 53 seniors in a junior-senior high school.

[c] A baseball team has played its schedule of 162 games.

[d] An archery set and a tennis racket together cost $48.

[e] Two bicycle riders start in opposite directions from a gasoline station. One rider travels at a speed of 8 mph.

[f] A certain number is added to 8, and this sum is divided by 6.

★ **4.** Make up three problems of your own similar to the ones suggested in Problem 3. Estimate the answers and give these problems to some of your classmates to solve.

Think

In the hexagons, arrange the digits 1 through 7 so that the sum of the digits is the same number in any three hexagons in a row.

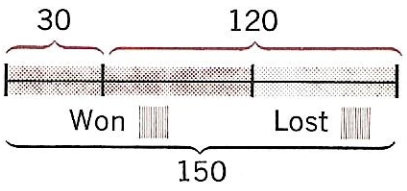
4-14 PROBLEM SOLVING

In working story problems, it is helpful to be able to compute rapidly and accurately. It is even more important to be able to decide what to do with the numbers given in a problem. It is impossible to memorize rules that will tell you how to work every problem you may need to solve. You simply must think carefully about the information given and then decide what operations to perform upon the numbers. The following examples show how you might reason in order to solve difficult problems.

A A baseball team has played 150 games, winning 30 more than it lost. How many has it won?

Solution 1. If the team had played 30 more games and lost them all, then it would have played 180 games and won half of them. Hence, it has won 90 games and lost 60.

Solution 2. We draw a line and think of it as 150 units long, representing the 150 games played. We mark off a part of the "Won" section to stand for the 30 extra games won. Of the 120 games left, we see that 60 were won and 60 lost. Hence, the team has won 90 games.



B A number is multiplied by 25 and to this product 17 is added. The final result is 2067. Find the original number.

Solution. Before adding 17, we must have had 2067 − 17 or 2050. Let us call the original number n; then $25 \cdot n = 2050$ and $n = 2050 \div 25 = 82$.

Exercises

It may be helpful to estimate answers before solving the problems. Draw diagrams if you feel they will help you.

1. An airplane averaged 340 mph for 9 hr. How far did it fly?
2. The distance between two cities is 2351 mi. A jet airplane has flown 854 mi from one city toward the other. How far is the airplane from the second city?
3. In five days of vacation driving, the Smith family drove 425, 513, 524, 411, and 387 mi. If their odometer reading was 23,496 when they left home, what was it after the five days of driving?
4. There are 1512 trees in a large orchard, with 36 trees in each row. How many rows are there?

5. The sum of the ages of two children is 14 years. What will be the sum of their ages 5 years from now?

6. A board 11 ft long is cut into two pieces. One piece is 3 ft shorter than the other. How long is each piece?

7. A car was driven 352 mi in 8 hr. What was its average speed in miles per hour?

8. A baseball team has lost 61 games. It has won 21 more games than it has lost. How many games has it played?

9. A baseball team has lost 54 games. It lost 5 of these games by only one run. If it had won these 5 games, it would have won and lost the same number of games. How many games has it played?

10. Points A, B, and C are in a straight line. The distance from A to B is 270 ft and that from B to C is 170 ft. Tell how far it is from A to C given that

[**a**] B is between A and C

[**b**] B is not between A and C

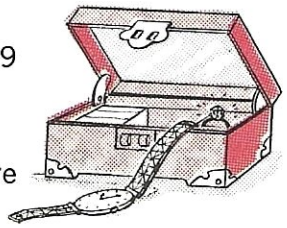
11. A class is divided into two groups. There are twice as many children in one group as in the other. Altogether, there are 24 children; how many are there in each group?

12. In a seventh-grade class there are three girls for every two boys. If there are 18 girls in the class, how many boys are there?

13. A boy went shopping with \$10 in his pocket. He spent half of his money in one store and \$1.70 in a second. He spent one third of the rest of his money in a third store. How much had he left?

14. A girl started shopping with a certain amount of money. She spent half of her money in one store and \$2.00 in a second. If she had \$4.00 left, how much money did she have at the beginning of her shopping trip?

15. I spent \$4.00 in one store and half of the remainder of my money in a second. If I have \$10 left, how much had I at first?

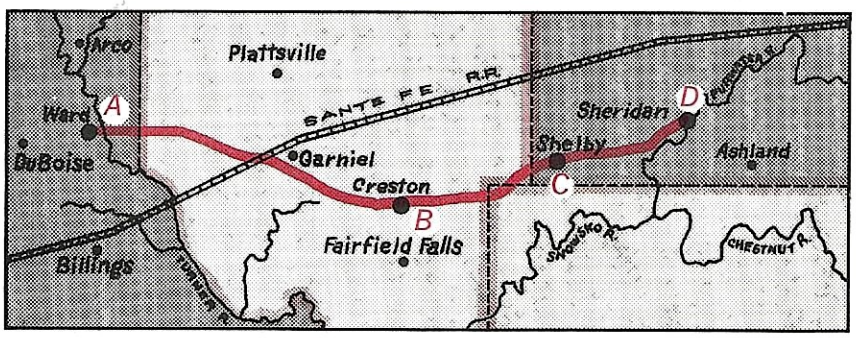


16. During one year, a city paid total salaries of \$165,300 to 29 firemen. What was the average salary for each fireman per year? About what was the average monthly salary?

17. A watch and ring cost together \$140. The watch cost \$30 more than the ring. What did each cost?

18. A watch cost three times as much as a ring. Together they cost \$140. What did each cost?

★**19.** In driving from Town *A* to Town *D*, you pass first through Town *B* and then through Town *C*. It is 10 mi farther from *A* to *B* than from *B* to *C*. It is also 10 mi farther from *B* to *C* than from *C* to *D*. If it is 390 mi from *A* to *D*, how far is it from *A* to *B*?



★**20.** In an orchard there are 7 peach trees for every 4 cherry trees. There are 60 more peach than cherry trees. How many cherry trees are there in the orchard?

★**21.** Find the largest number between 6900 and 7000 which when divided by 62 will give a remainder 30.

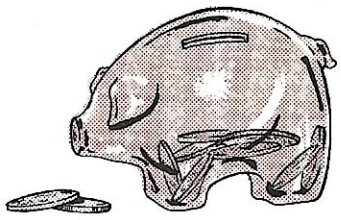
★**22.** Two cars start out together. The slower car averages 40 mph and the other car averages 60 mph. When the slower car has driven 280 mi, how much farther has the other car traveled?

★**23.** A first number is multiplied by a second, and the product is 12,802. When the first number is multiplied by a number one larger than the second, the product is 13,148. Determine the first and the second number.

★**24.** A first number is divided by a second one. The quotient is 40. When the first number is divided by a number one larger than the second, the quotient is 30. Find the first number.

Think

Cathy has 10 coins. None of the coins is a half dollar. The value of the coins is 59 cents. Can you decide what coins she must have?



CHAPTER REVIEW

Key Concepts

1. It is necessary to achieve a reasonable skill in simple computation with whole numbers.
2. It is necessary to understand why the methods used in addition, subtraction, multiplication, and division give correct answers.
3. **Exponents** are used to show how many times a number is to be used as a factor.

$$a^n = \underbrace{a \times a \times a \times a \cdots \times a}_{n \text{ factors}}$$

4. Multiplication can be thought of as repeated addition.
5. Division can be thought of as repeated subtraction.
6. The method of short division is useful when the divisor is a single-digit number.
7. The **arithmetic mean** or **average** of a set of numbers is found by adding the numbers and then dividing the sum by the number of addends.
8. In every division problem,
   dividend = (divisor · quotient) + remainder
9. The following are some suggestions for solving word problems.
   - Read carefully and logically
   - Select the important information
   - Make sketches and diagrams
   - Estimate answers
   - Decide what operations are needed
   - Compute correctly

Vocabulary List

addend, addition, arithmetic mean, average, difference, distributive principle, dividend, divisor, exponents, factor, perimeter, product, remainder, subtraction, sum

Review Exercises

1. Find the sums.

[a] $\begin{array}{r} 267 \\ 319 \\ +704 \\ \hline \end{array}$ [b] $\begin{array}{r} 31{,}286 \\ 51{,}495 \\ 73{,}354 \\ +21{,}621 \\ \hline \end{array}$ [c] $\begin{array}{r} 43{,}859 \\ 51{,}528 \\ 30{,}956 \\ +77{,}889 \\ \hline \end{array}$

2. Find the differences.

[a] $\begin{array}{r} 7488 \\ -3929 \\ \hline \end{array}$ [b] $\begin{array}{r} 93{,}000 \\ -78{,}289 \\ \hline \end{array}$ [c] $\begin{array}{r} \$18.98 \\ -7.99 \\ \hline \end{array}$

3. Find the products.

[a] $\begin{array}{r} 731 \\ \times 29 \\ \hline \end{array}$ [b] $\begin{array}{r} 237 \\ \times 548 \\ \hline \end{array}$ [c] $\begin{array}{r} 1001 \\ \times 431 \\ \hline \end{array}$

4. Find the quotients and remainders.

[a] $8\overline{)756}$ [c] $49\overline{)3862}$

[b] $23\overline{)684}$ [d] $7\overline{)42{,}184}$

5. Find the number for n.

[a] $17\overline{)n}$ $= 326$ [b] $n\overline{)473}$ $= 11$

[c] $61\overline{)33{,}855}$ $= n$

6. Solve the equations.

[a] $2 \cdot 4^2 = n$ [c] $2^2 + n^2 = 53$

[b] $3^2 \cdot 2^3 = n$ [d] $13^2 - n^2 = 12^2$

7. Cheryl checked the price of a certain article in four supermarkets. The chart shows how the prices compared. To the nearest cent, what was the average cost of the article?

Price (¢)
65
69
73
79

8. 236 is added to 928, and then the sum is doubled. If this result is divided by 6, what is the quotient?

9. Find a number which is 237 more than the product of 708 and 29.

10. In an orchard, there are 5 apple trees for every 3 peach trees. If there are 72 trees in all, how many are peach and how many are apple?

11. Football fields are 100 yd long from goal line to goal line. If it is 920 ft around a football field, how wide is the field?

12. The sum of seven numbers is 2143. If each number is increased by 50 and the seven resulting numbers are added, what will be the new sum?

★**13.** Two boys each work 8 hr and earn $20 all together. If one of the boys earns 20¢ more an hour than the other, how much did each boy earn?

★**14.** Two numbers are multiplied together. The product is 1204. Each of the two numbers is doubled. Give the product of these larger numbers.

★**15.** Find a number between 8400 and 8500 such that, upon dividing it by 73, we have a remainder of 60.

★**16.** A number is multiplied by 73, and the same number is also multiplied by 125. The smaller product is subtracted from the larger. The remainder is 12,532. Find the number.

CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 74–76.

———. *Magic House of Numbers.* New York, New American Library, 1957; pp. 67–79.

HOGBEN, LANCELOT. *The Wonderful World of Mathematics.* Garden City, New York, Doubleday and Company, 1955; pp. 62-69.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Computing Devices.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1961; pp. 1–36.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Short Cuts in Computing. Ibid.*, 1961; entire pamphlet.

KENYON, RAYMOND. *I Can Learn About Calculators and Computers.* Evanston, Illinois, Harper and Row, 1961; pp. 25–104.

SMITH, DAVID E. *History of Mathematics.* New York, Dover Publications, 1953; Vol. II, pp. 88–144.

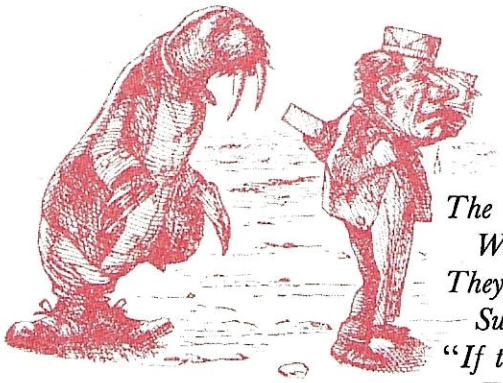
Activities

A. Learn how to operate a desk computer. Make up several addition, subtraction, and division problems and then work them yourself. Check your answers, using a desk computer.

B. A special method of computing was devised in the seventeenth century by a famous mathematician named John Napier. Find how to construct and use **Napier's Bones**.

C. Describe a method for subtracting from left to right instead of the usual right-to-left method.

D. Construct your own computer. (See *I Can Learn About Calculators and Computers*, pp. 74–104.)



The Walrus and the Carpenter
Were walking close at hand:
They wept like anything to see
Such quantities of sand:
"If this were only cleared away,"
They said, "it would be grand!"

"If seven maids with seven mops
Swept it for half a year,
Do you suppose," the Walrus said,
"That they could get it clear?"
"I doubt it," said the Carpenter,
And shed a bitter tear.

5

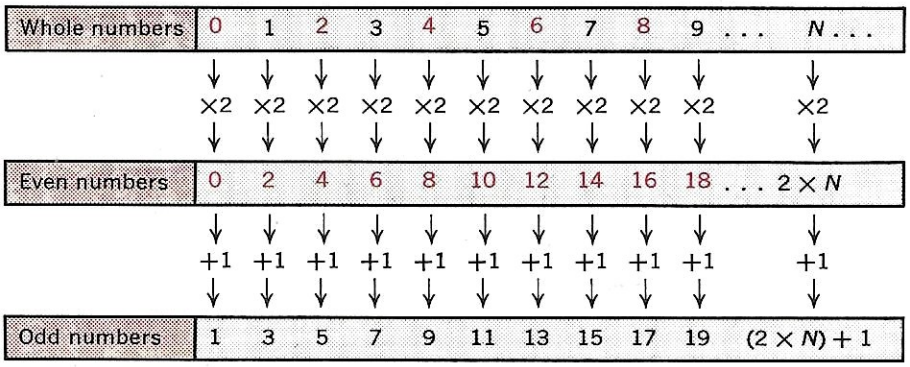
Number Theory

5-1 SPECIAL SETS OF WHOLE NUMBERS

Number theory is the study of properties of the set W of whole numbers.

$$W = \{0, 1, 2, 3, 4, \ldots\}$$

When we think of the whole numbers that have some special property, we think of a **subset** of W. For example, all the whole numbers with the property that they are **multiples** of 2 form a special subset of W.



A number that is 2 times a whole number is called an **even number**.

A number that is 1 more than an even number is called an **odd number**.

The set of even numbers: $E = \{0, 2, 4, 6, \ldots\}$

The set of odd numbers: $O = \{1, 3, 5, 7, \ldots\}$

Discussion Exercises

1. Consider the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Any part of S is a subset of S. Examples: $\{1, 2, 3\}$, $\{7, 2\}$, $\{6\}$
The set that has no elements (called the **empty set**) is a subset of S. Also, all of S is a subset of S.

[a] State the subset of all odd numbers in S.
[b] State the subset of all numbers in S less than 6.
[c] State the subset of all even numbers in S greater than 4.
[d] State the subset of all three-digit numbers in S.
[e] State the subset of all numbers in S less than 12.
★ [f] State the subset of all numbers in S that are less than 7 and greater than 3.

2. Give two special subsets of the whole numbers and give the first five numbers in each set.

Example: The set of whole numbers that are multiples of 3 is $T = \{0, 3, 6, 9, 12, \ldots\}$

Exercises

1. Give a numeral for each ▒.

[a] A whole number is even if ▒ is a factor of the number.
[b] A whole number is odd if ▒ is not a factor of the number.
[c] The set of odd numbers can be obtained by adding ▒ to every number in the set of even numbers.
[d] The set of even numbers can be obtained by subtracting ▒ from every number in the set of odd numbers.

2. Write **odd** or **even** to complete each sentence.

[a] The sum of any two even numbers is __?__.
[b] The sum of any two odd numbers is __?__.
[c] The sum of any even number and any odd number is __?__.
[d] The product of any two odd numbers is __?__.
[e] The product of any even number and any odd number is __?__.

3. The set of even numbers is **closed under addition** because the sum of any two even numbers is an even number.

[a] Is the set of odd numbers closed under addition; that is, is the sum of any two odd numbers an odd number?
[b] Is the set of even numbers closed under multiplication?
[c] Is the set of odd numbers closed under multiplication?

4. Copy and complete the addition and multiplication tables for "Evens" and "Odds."

+	E	O
E	E	
O		

×	E	O
E	E	
O		

5. Write **odd** or **even** to complete each sentence.

[a] The sum of any three odd numbers is _?_.
[b] The sum of any ten odd numbers is _?_.
[c] The product of any ten odd numbers is _?_.
★ [d] The sum of an even number of odd numbers is _?_.
★ [e] The sum of an odd number of odd numbers is _?_.

★ **6.** A student adds 20 numbers. Seven are odd and the rest are even. He gives as his answer 4366. What can you say about his work?

7. Work the problems in the "arithmetic of Odds and Evens."

[a] $E + E + O + E + E =$ ▒
[b] $O + E + O + E + O =$ ▒
[c] $O + O + O + O + O + O =$ ▒
★ [d] $O \cdot (E + O) + O \cdot (O + O) =$ ▒

★ **8.** Thirty-seven people are at a party. Some of them shake hands, but not everyone shakes hands with everyone else. Each person keeps track of the number of times he shakes hands. At the end of the evening, each person reports his number of handshakes, and all these numbers are added. Explain why the total will be even if everyone reports correctly.

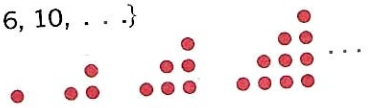
★ **9.** Show that in Exercise 8, if you counted the number of people who shook hands an odd number of times, you would get an even number.

Think

The ancient Greek mathematicians gave the name **triangular** numbers to the following set.

{1, 3, 6, 10, . . .}

Can you see why?



1. Give the next five triangular numbers.

2. Pick any triangular number. Add it to the next one. Do this several times. What do you notice?

5-2 FACTORS

The following examples explain what is meant when we say that one number is a factor of another.

(A) Since $5 \cdot 7 = 35$, we know that $\begin{cases} 7 \text{ is a factor of } 35. \\ 5 \text{ is a factor of } 35. \end{cases}$

(B) Since $6 \cdot 3 = 18$, we know that $\begin{cases} 6 \text{ is a factor of } 18. \\ 3 \text{ is a factor of } 18. \end{cases}$

If a, b, and c are whole numbers and if $a \cdot b = c$,

we know that $\begin{cases} a \text{ is a factor of } c. \\ b \text{ is a factor of } c. \end{cases}$

Every whole number, except the number 1, has at least two factors, and since $0 = 0 \cdot 0 = 1 \cdot 0 = 2 \cdot 0 = \cdots$, every whole number is a factor of zero.

$1 = 1 \cdot 1$	The only factor of 1 is 1.
$2 = 1 \cdot 2$	The factors of 2 are 1 and 2.
$3 = 1 \cdot 3$	The factors of 3 are 1 and 3.
$4 = 1 \cdot 4 = 2 \cdot 2$	The factors of 4 are 1, 2, and 4.
$5 = 1 \cdot 5$	The factors of 5 are 1 and 5.
$6 = 1 \cdot 6 = 2 \cdot 3$	The factors of 6 are 1, 2, 3, and 6.

Exercises

1. List all factors of the following numbers.

[a] 7 [b] 8 [c] 9 [d] 10 [e] 11 [f] 13

2. Use the equations to find all factors for each number.

[a] **25** $= 1 \cdot 25 = 5 \cdot 5$

[b] **77** $= 1 \cdot 77 = 7 \cdot 11$

[c] **143** $= 1 \cdot 143 = 11 \cdot 13$

[d] **27** $= 1 \cdot 27 = 3 \cdot 9$

[e] **24** $= 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$

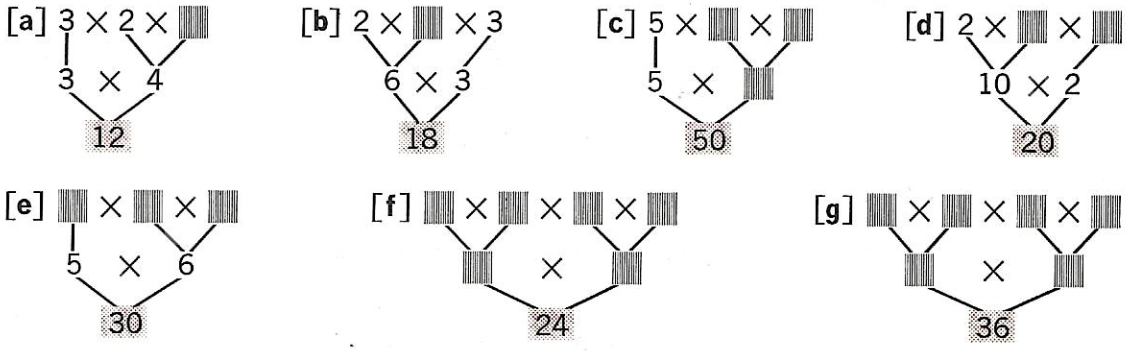
[f] **100** $= 1 \cdot 100 = 2 \cdot 50 = 4 \cdot 25 = 5 \cdot 20 = 10 \cdot 10$

3. The number 6 is the smallest number with four factors.

[a] Find the smallest number with five factors.

[b] Find the smallest number with six factors.

4. An interesting way to find some of the factors of a number is to build "factor trees." Copy the factor trees and supply the numbers for each ▮.



5. Show two more factor trees, different from those you gave in Exercise 4, for each of the numbers 30, 24, and 36.

6. List all factors of the following numbers.

[a] 36 [b] 80 [c] 72 [d] 25 [e] 50 [f] 100 [g] 200

★ **7.** [a] Find at least six numbers that have an odd number of factors.
[b] What do you think is true for all such numbers?

★ **8.** The sum of the factors of 6 is $6 + 6$.

$$1 + 2 + 3 + 6 = 6 + 6$$

The Greek mathematicians called 6 a **perfect** number. The next perfect number is between 25 and 30. What is it?

Think

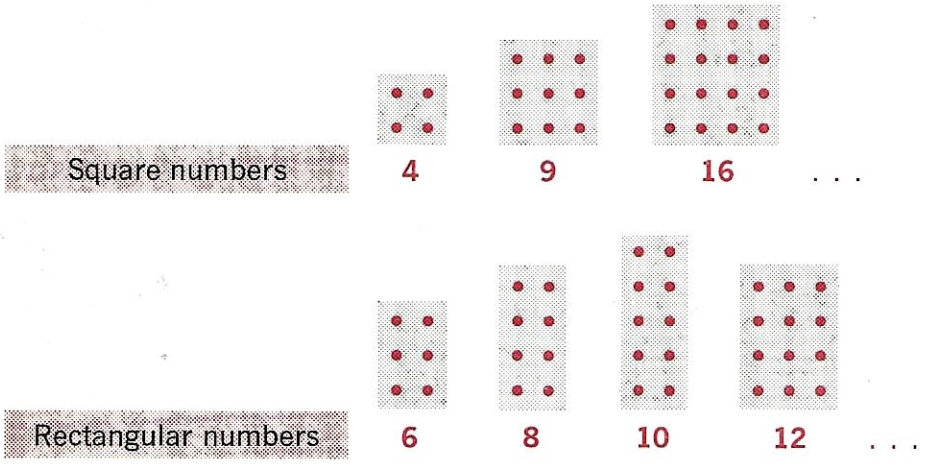
1. How many factors has 7? How many has $2 \cdot 7$?
2. How many factors has 9? How many has $2 \cdot 9$?
3. How many factors has 15? How many has $2 \cdot 15$?
4. How many factors has 45? How many has $2 \cdot 45$?
5. How many factors has 10? How many has $2 \cdot 10$?
6. How many factors has 14? How many has $2 \cdot 14$?
7. An **odd** number N has 100 factors.

   $\{1, \ldots, N\}$

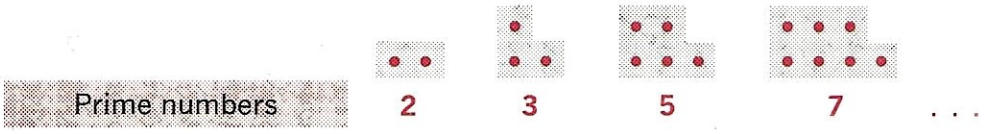
   How many factors has $2 \cdot N$?

5-3 PRIME NUMBERS

The ancient Greek mathematicians called some numbers **square** numbers and others **rectangular** numbers.



Some numbers cannot be pictured using square or rectangular arrays. Numbers greater than 1 that are neither square nor rectangular are called **prime** numbers. Each prime number has exactly two factors, itself and the number 1.



Square and rectangular numbers that are greater than 1 are called **composite numbers**. Each composite number can be expressed as the product of two smaller numbers. If the order of the factors is disregarded, each composite number can be expressed as a product of primes in **one and only one way**.

$12 = 2 \cdot 2 \cdot 3 \longrightarrow$ No set of primes other than $\{2, 2, 3\}$ has a product of 12.
$70 = 2 \cdot 5 \cdot 7 \longrightarrow$ No set of primes other than $\{2, 5, 7\}$ has a product of 70.
$30 = 2 \cdot 3 \cdot 5 \longrightarrow$ No set of primes other than $\{2, 3, 5\}$ has a product of 30.

When a composite number is written as a product of primes, we say that we have **factored the number completely** or have given the **prime factorization** of the number. The top row of a factor tree gives the prime factorization of a number.

Exercises

1. Copy and complete the equations so that you have the prime factorization of each given number.

[a] $6 = 2 \cdot a$
[b] $8 = 2 \cdot 2 \cdot b$
[c] $9 = 3 \cdot c$
[d] $10 = 2 \cdot d$
[e] $12 = 3 \cdot 2 \cdot e$
[f] $14 = 2 \cdot f$
[g] $15 = 3 \cdot g$
[h] $16 = 2 \cdot 2 \cdot 2 \cdot h$
[i] $18 = 2 \cdot 3 \cdot i$
[j] $20 = 2 \cdot 2 \cdot j$
[k] $21 = 3 \cdot k$
[l] $22 = 2 \cdot l$
[m] $24 = 2 \cdot 3 \cdot 2 \cdot m$
[n] $25 = 5 \cdot n$
[o] $26 = 2 \cdot o$
[p] $27 = 3 \cdot 3 \cdot p$
[q] $28 = 2 \cdot 2 \cdot q$
[r] $30 = 2 \cdot 3 \cdot r$

2. As in Exercise 1, factor completely each of the following numbers.

[a] 33 [b] 35 [c] 36 [d] 38 [e] 40 [f] 42 [g] 50 [h] 81 [i] 100 [j] 250 [k] 231 [l] 126

3. [a] Is 1 a prime number? [b] Is 1 a composite number? [c] Is zero a prime number? [d] Is zero a composite number?

4. List all the prime numbers less than 100.

5. Tell how many prime numbers are
[a] less than 10 [b] less than 100

6. Tell how many composite numbers are
[a] less than 10 [b] less than 100

7. There are 168 prime numbers less than 1000. How many composite numbers are there less than 1000?

8. How many of the prime numbers are even numbers?

9. There is just one pair of prime numbers whose difference is 1. What is the pair?

10. The smallest pair of prime numbers with a difference of 2 is 3 and 5. Such numbers are called **twin primes**. Find six other pairs of twin primes.

11. [a] Find two prime numbers whose difference is 3.
[b] Do you think there is any other such pair?

12. Note that 11 is a prime number: $11 + 2 = 13$ is prime; $13 + 4 = 17$ is prime; $17 + 6 = 23$ is prime. How long does this pattern continue?

★ **13.** For each number below, a factorization is given. Decide whether or not it is a prime factorization. If not, factor the number completely.

[a] $72 = 2 \cdot 2 \cdot 2 \cdot 9$ [c] $713 = 23 \cdot 31$ [e] $1331 = 11 \cdot 121$ [g] $1001 = 77 \cdot 13$
[b] $216 = 8 \cdot 3 \cdot 3 \cdot 3$ [d] $627 = 11 \cdot 57$ [f] $360 = 2^3 \cdot 45$ [h] $6923 = 161 \cdot 43$

★ **14.** Use the facts $2 \cdot 5 = 10$, $2^2 \cdot 5^2 = 100$, $2^3 \cdot 5^3 = 1000$, and $2^4 \cdot 5^4 = 10{,}000$ to find the products quickly.

[a] $2 \cdot 7 \cdot 5$ [d] $2^3 \cdot 11 \cdot 5^2$ [g] $2^4 \cdot 23 \cdot 5^4$ [j] $2^5 \cdot 7 \cdot 11 \cdot 5^4$
[b] $2^2 \cdot 13 \cdot 5^2$ [e] $2^2 \cdot 12 \cdot 5^3$ [h] $2^4 \cdot 71 \cdot 5^3$ [k] $2^4 \cdot 3 \cdot 7 \cdot 5^4$
[c] $2^2 \cdot 45 \cdot 5^2$ [f] $2^3 \cdot 147 \cdot 5^3$ [i] $2^3 \cdot 101 \cdot 5^4$ [l] $2^6 \cdot 3^2 \cdot 5^6$

★ **15.** Use the fact that $7 \cdot 11 \cdot 13 = 1001$ to find the products quickly.

[a] $7^2 \cdot 11 \cdot 13$ [c] $7 \cdot 11 \cdot 13^2$ [e] $7 \cdot 11 \cdot 13 \cdot 17$ [g] $7 \cdot 11^2 \cdot 13 \cdot 23$
[b] $7 \cdot 11^2 \cdot 13$ [d] $7^2 \cdot 11^2 \cdot 13$ [f] $7 \cdot 11 \cdot 13 \cdot 53$ [h] $7^2 \cdot 11^2 \cdot 13^2$

★ **16.** Use the fact that $7 \cdot 11 \cdot 13 = 1001$ to find the prime factorization of the following numbers.

[a] 3003 [c] 25,025 [e] 255,255 [g] 100,100
[b] 8008 [d] 84,084 [f] 125,125 [h] 999,999

★ **17.** Jim saw a car with the license number 673,673. "That's an interesting number," he said. "If you divide it by 7, 11, 13, or 77, the remainder is zero." Later, he saw a car with license number 718,723. "That number is just as interesting," he said. "If you divide it by 7, 11, 13, or 77, the remainder each time is 5." Explain Jim's trick.

Think

No even number after 2 is a prime number. But

Every even number larger than 2 seems to be the sum of two primes. No one has ever been able to prove that this is so, but most mathematicians believe it.

1. Check this for the even numbers up to 50.
2. Express 200 as the sum of two primes.

$4 = 2 + 2$ $16 = 5 + 11$
$6 = 3 + 3$ $18 = 7 + 11$
$8 = 3 + 5$ $20 = 7 + 13$
$10 = 3 + 7$ ·
$12 = 5 + 7$ ·
$14 = 3 + 11$ ·

5-4 GREATEST COMMON FACTOR (GCF)

Often, you need to know the largest number that is a **common factor** of two or more numbers. When you want to express a fraction like $\frac{12}{18}$ in lower terms, you hunt for a common factor of 12 and 18.

(A) You can use 2.

$$\frac{12}{18} = \frac{2 \cdot 6}{2 \cdot 9} = \frac{6}{9}$$

(B) You can use 3.

$$\frac{12}{18} = \frac{3 \cdot 4}{3 \cdot 6} = \frac{4}{6}$$

(C) You can use 6.

$$\frac{12}{18} = \frac{6 \cdot 2}{6 \cdot 3} = \frac{2}{3}$$

In Examples A and B, the lowest-terms fraction was not obtained. In Example C, the lowest-terms fraction was obtained. While 2, 3, and 6 are all common factors of 12 and 18, 6 is the **greatest common factor** (GCF) cf 12 and 18.

(1) Set of factors of 18 $\longrightarrow A = \{1, 2, 3, 6, 9, 18\}$

(2) Set of factors of 12 $\longrightarrow B = \{1, 2, 3, 4, 6, 12\}$

(3) Set of common factors of 18 and 12 $= \{1, 2, 3, 6\}$

The GCF of 18 and 12 is 6.

The set of all numbers that are in both set A and set B is called the **intersection** of sets A and B. We write, $A \cap B = \{1, 2, 3, 6\}$. We read $A \cap B$ as "the intersection of sets A and B."

Exercises

1. [a] List the factors of 12.
[b] List the factors of 20.
[c] List the common factors of 12 and 20.
[d] What number is the GCF of 12 and 20?

2. [a] List the factors of 10.
[b] List the factors of 30.
[c] List the common factors of 10 and 30.
[d] What number is the GCF of 10 and 30?

3. [a] List the factors of 18.
[b] List the factors of 27.
[c] List the common factors of 18 and 27.
[d] What number is the GCF of 18 and 27?

4. [a] List the factors of 28.
[b] List the factors of 35.
[c] List the common factors of 28 and 35.
[d] What number is the GCF of 28 and 35?

5. Find the GCF of each pair of numbers. (Sometimes it is 1.)

[a] 10, 25	[f] 8, 27	[k] 11, 12	[p] 85, 68
[b] 6, 18	[g] 8, 18	[l] 14, 35	[q] 45, 36
[c] 8, 15	[h] 15, 25	[m] 100, 60	[r] 90, 120
[d] 24, 36	[i] 6, 8	[n] 75, 125	[s] 150, 270
[e] 40, 100	[j] 12, 16	[o] 48, 72	[t] 450, 360

6. [a] Which pairs of numbers in Exercise 5 have just one common factor and what is it?
[b] What number is a common factor of every pair of numbers?

7. Find the GCF of each pair of numbers. (The complete factorization is given and may help you. The answer to Part a is 14.)

[a] $42 = 2 \cdot 3 \cdot 7$ $70 = 2 \cdot 5 \cdot 7$	[d] $12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot 3 \cdot 3$	[g] $42 = 3 \cdot 7 \cdot 2$ $24 = 3 \cdot 2 \cdot 2 \cdot 2$	[j] $90 = 2 \cdot 3^2 \cdot 5$ $60 = 2^2 \cdot 3 \cdot 5$
[b] $70 = 2 \cdot 5 \cdot 7$ $30 = 2 \cdot 3 \cdot 5$	[e] $10 = 2 \cdot 5$ $70 = 2 \cdot 5 \cdot 7$	[h] $24 = 3 \cdot 2^3$ $6 = 2 \cdot 3$	[k] $105 = 3 \cdot 5 \cdot 7$ $225 = 3^2 \cdot 5^2$
[c] $55 = 5 \cdot 11$ $15 = 3 \cdot 5$	[f] $18 = 2 \cdot 3 \cdot 3$ $15 = 3 \cdot 5$	[i] $40 = 5 \cdot 2^3$ $10 = 5 \cdot 2$	[l] $385 = 5 \cdot 7 \cdot 11$ $175 = 5^2 \cdot 7$

★ **8.** The letter N represents a number between 50 and 60. The GCF of N and 16 is 8. Find the number.

★ **9.** Thirteen is not a factor of N. What is the GCF of 13 and N?

★ **10.** The GCF of A and B is 5. What is the GCF of $2 \cdot A$ and $2 \cdot B$?

★ **11.** Two numbers have a GCF of 12. One number is 5 times as large as the other. Find the two numbers.

Think

The Greek mathematicians called 12 an **abundant** number because the sum of the factors of 12 is greater than 12 + 12.

$$1 + 2 + 3 + 4 + 6 + 12 > 12 + 12$$

The number 12 is also the smallest abundant number. Find the next two abundant numbers.

5-5 LEAST COMMON MULTIPLE (LCM)

The **least common multiple** (LCM) of two numbers is the smallest number (other than zero) that is a multiple of both numbers. You use this concept in the addition of rational numbers when you need to choose a common denominator for two or more fractions.

Consider $\frac{1}{4} + \frac{1}{6}$.

(A) You can choose 24.

$$\frac{1}{4} + \frac{1}{6} = \frac{6}{24} + \frac{4}{24} = \frac{10}{24}$$

(B) You can choose 12.

$$\frac{1}{4} + \frac{1}{6} = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$$

In Example A, the least common denominator (LCD) was not obtained. In Example B, the LCD was obtained. Your work with addition and subtraction of rational numbers is often simpler if you choose the LCD. You can do this by finding the LCM of the two denominators.

(1) Multiples of 4 $\longrightarrow A = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36, \ldots\}$

(2) Multiples of 6 $\longrightarrow B = \{0, 6, 12, 18, 24, 30, 36, 42, \ldots\}$

(3) Common nonzero multiples of 4 and 6 $\longrightarrow A \cap B = \{12, 24, 36, \ldots\}$

The LCM of 4 and 6 is 12.

You may find it easier just to list multiples of the larger number until you find a number that is a multiple of the smaller.

Example: Find the LCM of 8 and 6.
Multiples of 8 $\longrightarrow \{0, 8, 16, 24\}$.
The LCM of 8 and 6 is 24.

Exercises

1. [a] List multiples of 3 to 36. [b] List multiples of 4 to 36.
[c] List the common multiples of 3 and 4 to 36.
[d] Pick out the LCM of 3 and 4.

2. For each pair of numbers, give the first six nonzero multiples of each number. Then give the LCM of the two numbers.

Example: For 3 and 7, write:
$\{3, 6, 9, 12, 15, 21\}$, $\{7, 14, 21, 28, 35, 42\}$.
The LCM of 3 and 7 is 21.

[a] 2, 5 [b] 2, 4 [c] 4, 6 [d] 6, 9 [e] 6, 10 [f] 4, 5

3. [a] List the multiples of 2 that are less than 25.
[b] List the multiples of 3 that are less than 25.
[c] List the common multiples of 2 and 3 that are less than 25.
[d] What is the LCM of 2 and 3?
[e] Is the LCM of 2 and 3 a factor of every common multiple of 2 and 3?

4. For each pair of numbers, find the LCM by finding the smallest multiple of the larger number that is also a multiple of the smaller number.

Example: 8, 10; Multiples of 10 are {10, 20, 30, 40}.
↑
40 is also a multiple of 8.

[a] 3, 5	[e] 8, 12	[i] 7, 10	[m] 4, 6
[b] 10, 4	[f] 8, 3	[j] 10,11	[n] 40, 60
[c] 2, 7	[g] 5, 6	[k] 15, 25	[o] 12, 15
[d] 6, 12	[h] 1, 15	[l] 16, 24	[p] 120, 150

5. If you were working these addition problems, what common denominator would you choose? Think about the LCM of the denominators.

[a] $\frac{1}{2}+\frac{3}{4}$	[c] $\frac{3}{5}+\frac{1}{2}$	[e] $\frac{7}{12}+\frac{1}{8}$	[g] $\frac{7}{10}+\frac{3}{8}$	[i] $\frac{7}{12}+\frac{1}{16}$
[b] $\frac{2}{3}+\frac{1}{2}$	[d] $\frac{3}{8}+\frac{5}{6}$	[f] $\frac{4}{3}+\frac{2}{5}$	[h] $\frac{5}{8}+\frac{3}{16}$	[j] $\frac{4}{15}+\frac{3}{20}$

6. To find the LCM of 4, 6, and 10, we can first find the LCM of 6 and 10. It is 30. Now find the LCM of 30 and 4. It is 60. Thus 60 is the LCM of 4, 6, and 10. Find the LCM of the following groups of numbers.

[a] 2, 3, 5	[c] 2, 4, 6	[e] 4, 6, 12	[g] 3, 4, 8	[i] 10, 20, 30
[b] 2, 3, 4	[d] 2, 4, 8	[f] 2, 3, 6	[h] 5, 10, 15	[j] 8, 12, 15

7. Is there any prime number that is
[a] a multiple of 2? [b] a multiple of 3? [c] a multiple of 6?

8. Suppose you know that 15 is a factor of a number N. Find three other numbers that are factors of N.

9. If 6 is a factor of M, what two prime numbers are factors of M?

10. What prime numbers are factors of each of the following products?

[a] $2 \cdot 5$ [b] $2^2 \cdot 5^2$ [c] $3 \cdot 7$ [d] $3^2 \cdot 7^3$ [e] $2 \cdot 3 \cdot 5$ [f] $7^2 \cdot 3 \cdot 5^4$

11. [a] Is 5 a factor of $7 \cdot 5$?
[b] Is 3 a factor of $7 \cdot 5$?
[c] Is 11 a factor of $2 \cdot 7$?
[d] Is 7 a factor of $3^2 \cdot 7^2$?
[e] Is 5 a factor of $3^2 \cdot 7^2$?
[f] Is 3 a factor of $5 \cdot 7^2$?

12. [a] Is 3 a factor of $2^2 \cdot 5^2$?
[b] Is $2 \cdot 3$ a factor of $2^2 \cdot 5^2$?
[c] Is $2 \cdot 10$ a factor of $2^2 \cdot 5^2$?
[d] Is $2 \cdot 7$ a factor of $2^2 \cdot 5^2$?

★ **13.** Solve the equations.
[a] $2^3 = 2^2 \cdot a$
[b] $3^4 = 3^2 \cdot b$
[c] $2^2 \cdot 3 = 2^2 \cdot c$
[d] $3^3 \cdot 7 \cdot 11 = 3^3 \cdot 7 \cdot d$
[e] $3^2 \cdot 2^2 = 3 \cdot 2 \cdot e$
[f] $3^3 \cdot 5^2 = 3^2 \cdot 5 \cdot f$
[g] $3^3 \cdot 5^2 \cdot 7 = 3 \cdot 5^2 \cdot g$
[h] $2^4 \cdot 3^3 \cdot 5^2 = 2^2 \cdot 3^2 \cdot h$
[i] $3^3 \cdot 5^3 \cdot 7^3 \cdot 11^3 = 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot i$
[j] $2^5 \cdot 3^4 \cdot 5^4 \cdot 13^3 = 2^2 \cdot 3^3 \cdot 5^4 \cdot 13 \cdot j$

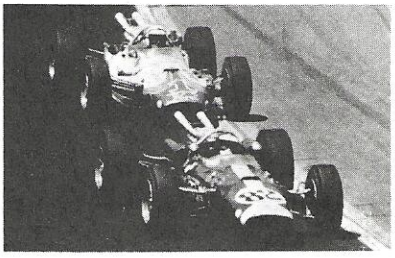
★ **14.** Show that
[a] $5^3 \cdot 7^3$ is a multiple of $5^2 \cdot 7^3$
(**Hint:** Find the number a so that $5^3 \cdot 7^3 = 5^2 \cdot 7^3 \cdot a$.)
[b] $7^3 \cdot 11^2$ is a multiple of $7^2 \cdot 11$
[c] $5^3 \cdot 7^2 \cdot 17$ is a multiple of $5^2 \cdot 7^2 \cdot 17$
[d] $7^4 \cdot 11^3 \cdot 19^2$ is a multiple of $7^3 \cdot 11 \cdot 19$

★ **15.** Choose a multiple of $2^2 \cdot 3^2$ that is also a multiple of the following numbers.
[a] 2^3 [b] 3^3 [c] $2^2 \cdot 5$ [d] $2 \cdot 3^3$ [e] $3 \cdot 5$ [f] $3^3 \cdot 5$

★ **16.** [a] $2^3 \cdot 3^3$ is a common multiple of $2^2 \cdot 3$ and $2 \cdot 3^3$. Find a smaller common multiple .
[b] $3^4 \cdot 5^4 \cdot 7^4$ is a common multiple of $3^3 \cdot 5^3 \cdot 7^3$ and $3 \cdot 5^4 \cdot 7$. Find a smaller common multiple.

★ **17.** Find the LCM of each number pair.
[a] 2^2 and 3^2
[b] 3^3 and 5^2
[c] 2^3 and $2 \cdot 3^2$
[d] $2^2 \cdot 5$ and $3 \cdot 5^2$
[e] $2^2 \cdot 3$ and $3^2 \cdot 5$
[f] $3^2 \cdot 5^3 \cdot 7$ and $3^3 \cdot 5 \cdot 11$

★ **18.** Two cars are driving around a 2-mi track. They start off together. Car A makes one lap every 80 sec. Car B makes one lap every 60 sec.
[a] How long will it take the faster car to gain a lap on the slower one?
[b] How long will it take the faster car to gain a lead of 10 laps?



CHAPTER REVIEW

Key Concepts

1. Number theory is the study of properties of sets of whole numbers.
2. Of special interest are certain special subsets of whole numbers such as **even**, **odd**, and **square** numbers.
3. Many interesting properties of numbers are related to their factors. For example, each prime number has exactly two factors, itself and the number 1.
4. The prime numbers can be thought of as basic building blocks for forming other numbers by multiplication.
5. Each composite number has just one prime factorization.
6. The ideas of **greatest common factor** (GCF) and **least common multiple** (LCM) are important in working with fractions.
7. The GCF of two numbers is the largest number that is a factor of both numbers.
8. The LCM of two numbers is the smallest number that is a multiple of both numbers.
9. We can determine the GCF (or LCM) of two numbers by finding the prime factorization of each number.

Vocabulary List

closure
common factor
common multiple
complete factorization
composite number
even number
factor
greatest common factor
intersection of sets
least common multiple
multiple
odd number
prime factorization
prime number
rectangular number
square number
subset
triangular number

Review Exercises

1. Give a definition for
   [a] an even number [b] an odd number
2. $A + B$ is odd. A is odd. What can you say about B?
3. $A \cdot B$ is even. A is odd. What can you say about B?
4. Let N represent the sum of 31 numbers. Twenty of these numbers are even numbers. The rest of the numbers are odd. Is N an even or an odd number?
5. The set of all multiples of 5 is
   $\{0, 5, 10, 15, 20, 25, \ldots\}$
   Is this set
   [a] closed under addition?
   [b] closed under multiplication?
6. The set of all powers of 10 is
   $\{10^0 = 1, 10^1 = 10, 10^2, 10^3, 10^4, \ldots\}$
   Is this set
   [a] closed under addition?
   [b] closed under multiplication?
7. Which of the following are prime numbers?
   [a] 3 [c] 35 [e] 83
   [b] 13 [d] 51 ★[f] 121
8. Which of the following are composite numbers? Of these, which are square numbers?
   [a] 2 [c] 25 [e] 81
   [b] 18 [d] 63 ★[f] 441
9. List all factors of
   [a] 14 [b] 60 [c] 210
10. Show that 48 is a factor of 5808.
11. Find a number between 20 and 30 that has an odd number of factors.
12. Give the prime factorization of 84.
13. Replace each factorization by a prime factorization.
    [a] $72 = 8 \cdot 9$ [c] $100 = 4 \cdot 25$
    [b] $144 = 16 \cdot 9$ [d] $675 = 5 \cdot 135$

14. If A is the set of factors of 135 and B is the set of factors of 175, list all numbers in

[a] set A [b] set B [c] set $(A \cap B)$

15. Find the GCF of

[a] 8, 16 [c] 12, 20 [e] 60, 18
[b] 8, 15 [d] 14, 49 [f] 26, 182

16. Find a pair of prime numbers whose difference is

[a] 10 [b] 22 [c] 100

17. Find the LCM of 6 and

[a] 3 [b] 9 [c] 8 [d] 4 [e] 5

18. Use the prime factorization to find the GCF for each pair of numbers.

[a] $15 = 3 \cdot 5$; $25 = 5 \cdot 5$
[b] $44 = 2^2 \cdot 11$; $52 = 2^2 \cdot 13$
[c] $77 = 7 \cdot 11$; $143 = 11 \cdot 13$
[d] $81 = 3^4$; $72 = 2^3 \cdot 3^2$

19. Find the LCM of each group.

[a] 4, 8, 12 [c] 4, 6, 9
[b] 3, 6, 10 [d] 10, 12, 20

★ **20.** In hunting for the GCF of 105 and N, we find that 3 is not a factor of N and 5 is not a factor of N. We know that $105 = 3 \cdot 5 \cdot 7$. What can we say about the GCF of 105 and N?

CHAPTER SUPPLEMENT

Reading List

CUTLER, A., and MCSHANE (trans.). *The Trachtenberg Speed System of Basic Mathematics.* Garden City, New York, Doubleday and Company, 1960; entire book.

GAMOW, GEORGE. *One, Two, Three, Infinity.* New York, Viking Press, 1961; pp. 3–38.

GARDNER, MARTIN. *The Scientific American Book of Mathematical Puzzles and Diversions.* New York, Simon and Schuster Inc., 1959; pp. 15–22.

HEATH, ROYAL. *Mathemagic.* New York, Dover Publications, 1953; pp. 7–31.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Fun with Mathematics.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1960; entire pamphlet.

———. *Number Patterns. Ibid.,* pp. 1–30.

REID, CONSTANCE. *From Zero to Infinity.* New York, Crowell Company, 1960; pp. 1–56.

SCHOOL MATHEMATICS STUDY GROUP. *Mathematics for Junior High School.* New Haven, Yale University Press, 1961 (Revision); Vol. 1, Part I, pp. 151–160, 164–169, 178–188.

Activities

A. Find a definition of **prime number** other than the one used in this text. Discuss any differences in the definitions.

B. When a first number is a **multiple** of a second number, then the first number is said to be **divisible** by the second. Explain the simple shortcut for deciding when a number is divisible by 2, by 5, by 3, by 9. Now try by 4, by 8, by 27.

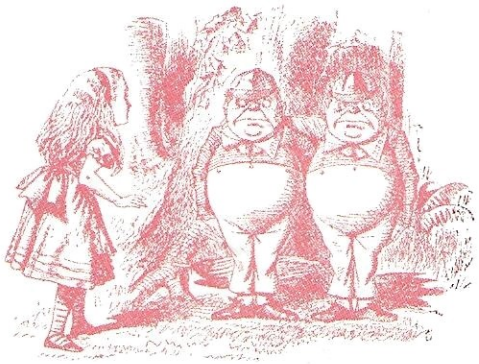
C. Look up the **Sieve of Eratosthenes** and use the method of the sieve to find the primes less than 100.

D. Find a number puzzle to present to your class. Be ready to explain why the puzzle works as it does. (See *Fun with Mathematics.*)

E. Find out what the Greek mathematicians meant when they said that two numbers were an **amicable** pair. Find two such pairs of numbers.

F. Describe and discuss **Goldbach's Conjecture.**

G. Find material that discusses the fact that there is no last, or largest, prime. Write a paper on your findings.



TWEEDLEDUM AND TWEEDLEDEE

They were standing under a tree, each with an arm round the other's neck, and Alice knew which was which in a moment, because one of them had "DUM" embroidered on his collar, and the other "DEE."

6

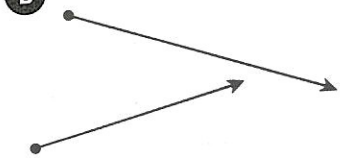
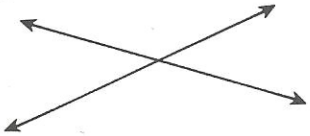
Geometry Unit II
Congruent Angles and Triangles

A. NONMETRIC GEOMETRY

6-1 CONGRUENT SETS

The following story will help you understand an important geometric concept: Joe drew a triangle. He then drew a copy of this figure that was so good that he could not tell it from the original.

Joe's two drawings illustrate what is meant by **congruent** geometric figures. It may help you to think of congruent figures (or sets of points) as figures that look exactly alike.

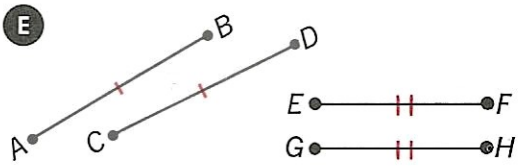
A	B	C
		
Any two **points** are congruent to each other.	Any two **rays** are congruent to each other.	Any two **lines** are congruent to each other.

Given certain conditions, we can describe other pairs of figures that are congruent.

D

Two pairs of points are congruent if they are equally far apart.

E

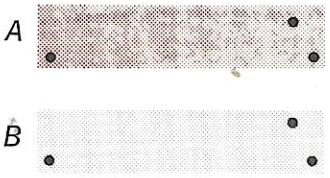


$\overline{AB} \cong \overline{CD}$ $\overline{EF} \cong \overline{GH}$

Two segments are congruent if their endpoints are equally far apart.

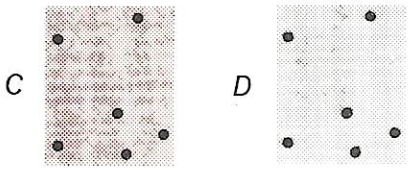
Using the basic ideas of congruence, we can investigate more complicated examples of congruent figures.

F



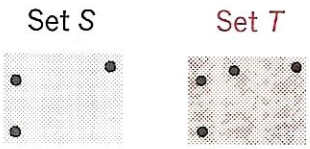
The set of three points in *A* is congruent to the set of three points in *B*.

G



The set of six points in *C* is congruent to the set of six points in *D*.

Discussion Exercises



1. [a] Explain why set *S* is not congruent to set *T*.

[b] Where would you put a fourth point in set *S* in order to make *S* and *T* congruent?

2. Jack said, "To tell whether or not two sets are congruent, I just think about placing one set on top of the other to see if they fit."

[a] Explain what Jack meant.

[b] Must two sets of points have the same number of points in order to be congruent?

[c] Is it possible for two sets to have the same number of points and yet not be congruent? Explain your answer.

3. When the objects in two sets are matched one to one, we often say that the matched objects **correspond** to each other. When two sets of points are congruent, we can find a correspondence by thinking about placing one set over the other. Suppose the sets on the right are congruent.

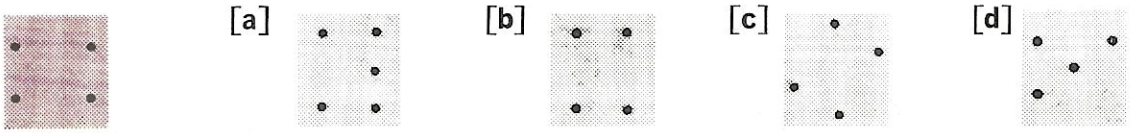
[a] If point *A* corresponds to *E* and point *B* to *F*, to what point does point *C* correspond?

[b] Is it true that the point pair *C*, *D* is congruent to the point pair *G*, *H*?

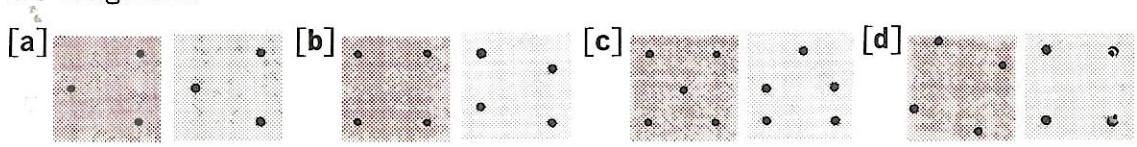
[c] Is point pair *A*, *D* congruent to point pair *F*, *G*? Explain your answer.

Exercises

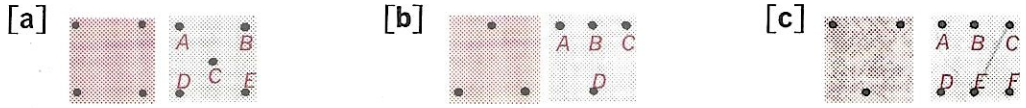
1. Which two sets are not congruent to the colored set?



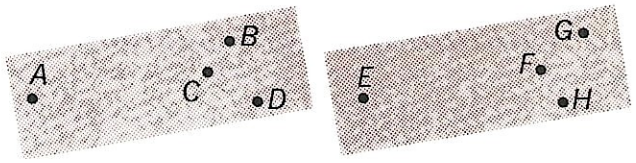
2. For each part, tell whether or not you think the two sets of points are congruent.



3. Find a subset of points of the second set (the gray set) that is congruent to the points in the first set. In Parts **c** through **g**, find two subsets.

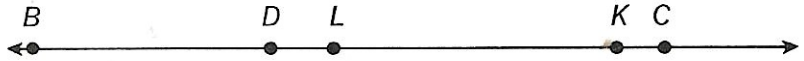


4. Two sets of four points are congruent. Points A and E correspond to each other. What other pairs of points correspond?

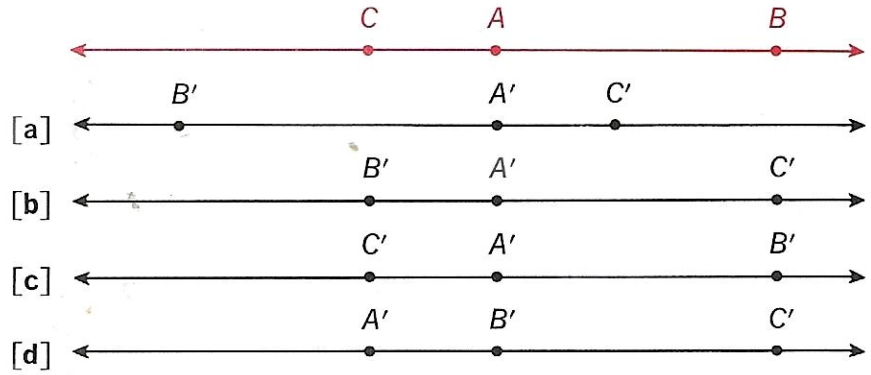


5. Suppose three points A, B, and A' are given. Where are all possible locations for the point B' so that $\overline{AB} \cong \overline{A'B'}$?

6. Draw a ray. Mark on it points B', C', D', K', L' corresponding to the points shown on the following ray so that the set $\{B, C, D, K, L\}$ is congruent to the set $\{B', C', D', K', L'\}$.

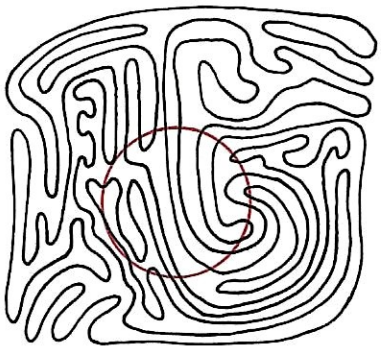


7. For each part, tell whether or not the set of points $\{A', B', C'\}$ is congruent to the set of points $\{A, B, C\}$.



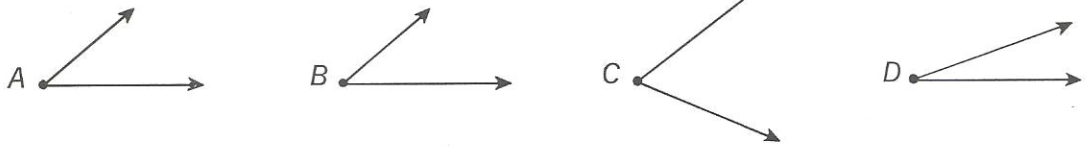
Think

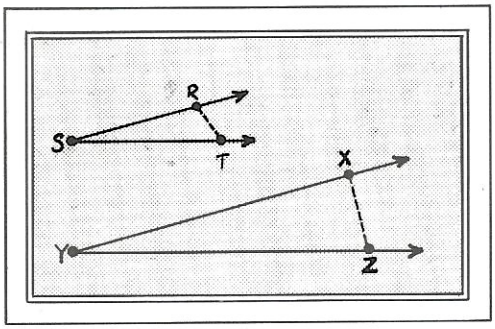
With tracing paper, copy this figure. It is a **simple closed curve.** (Think of a loop of string that does not cross itself.) Within the colored circle, shade the interior of this simple closed curve.



6-2 CONGRUENT ANGLES

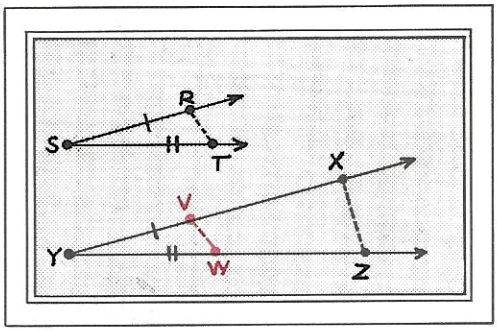
A group of students were comparing angles. Everyone agreed that $\angle A$ and $\angle B$ were about the same size. They also agreed that $\angle C$ was larger than $\angle D$.

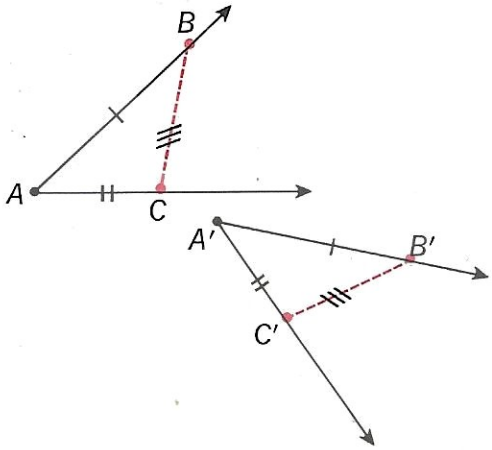




But the students disagreed when they compared $\angle RST$ with $\angle XYZ$. Gary said that $\angle XYZ$ was larger than $\angle RST$ because it "opened up" more. He pointed out that $\overline{XZ} > \overline{RT}$. Sue said that Gary should not compare $\overline{XZ}$ with $\overline{RT}$ because $\overline{XZ}$ was "farther out" from the vertex than $\overline{RT}$.

Sue went to the chalkboard and drew this picture. She said, "If I make $\overline{YW} \cong \overline{ST}$ and $\overline{YV} \cong \overline{SR}$, then it's fair to compare $\overline{RT}$ with $\overline{VW}$ to decide how the angles compare." Using this new picture, everyone in class agreed that the two angles would be congruent if $\overline{RT}$ were congruent to $\overline{VW}$.

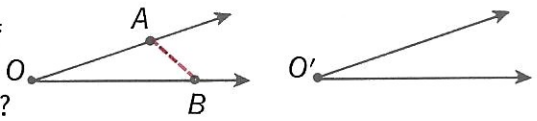




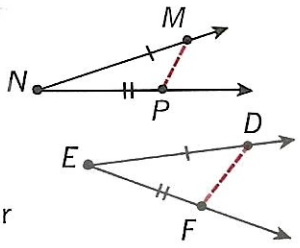
If points A, B, and C are congruent to points A', B', and C' and if A corresponds to A', then $\angle A$ is congruent to $\angle A'$. We write: $\angle A \cong \angle A'$.

Discussion Exercises

1. Suppose you want to compare $\angle O$ and $\angle O'$. Where on the sides of $\angle O'$ will you select points so that you can use $\overline{AB}$ for your comparison?



2. Use the figures to explain how you would tell someone why $\angle MNP$ is smaller than $\angle DEF$.



3. In the figures for Exercise 2, let us call points M and D corresponding points, since they are equally far from their respective vertices.

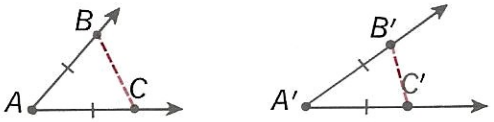
   [a] What other pair of points correspond?

   [b] If two points on one angle (one on each ray) are closer together than their corresponding points on another angle, what can you say about the two angles? Explain your answer.

   [c] If two angles are congruent, what can you say about pairs (one on each ray) of corresponding points on the two angles?

Exercises

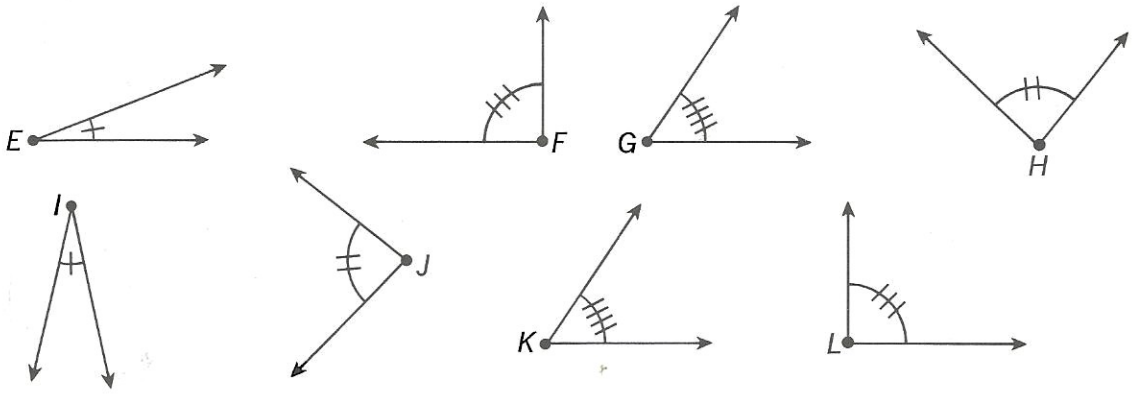
1. In the figures, $\overline{BC} > \overline{B'C'}$. How do $\angle A$ and $\angle A'$ compare?



2. Suppose that $\angle 1 \cong \angle 2$. If M, N and P, Q are corresponding point pairs on the two angles, what can you say about $\overline{MN}$ and $\overline{PQ}$?

3. We often use slash marks to indicate that two angles are congruent. The same number of slashes indicates that the two angles are congruent.
   Examples: $\angle A \cong \angle B$ and $\angle C \cong \angle D$

Using the following figures, pick out the angles that are congruent and write the congruence as in the preceding examples.



4. One circle has a center at M and the other circle has a center at H. Both circles have the same radius. Give the correct symbol ($<$, $>$, or $\cong$) for each ▨.

[a] If $\overline{LN} \cong \overline{GI}$, then $\angle M$ ▨ $\angle H$.

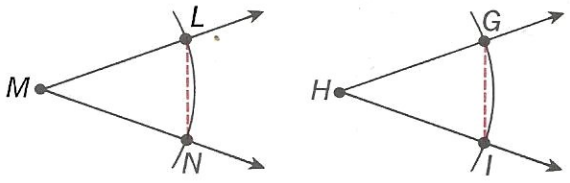
[b] If $\overline{LN} > \overline{GI}$, then $\angle M$ ▨ $\angle H$.

[c] If $\overline{LN} < \overline{GI}$, then $\angle M$ ▨ $\angle H$.

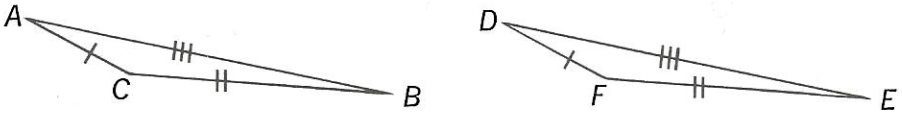
[d] If $\overline{LN} \cong \overline{GI}$, then $\angle LMN$ ▨ $\angle GHI$.

[e] If $\overline{LN} \cong \overline{GI}$, then $\angle LNM$ ▨ $\angle GIH$.

[f] If $\overline{LN} \cong \overline{GI}$, then $\angle NLM$ ▨ $\angle IGH$.



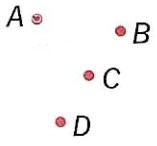
5. The sets $\{A, B, C\}$ and $\{D, E, F\}$ are congruent. Copy and complete each statement.



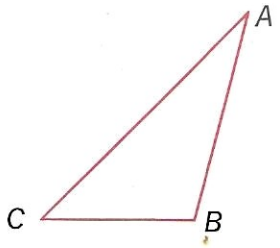
[a] $\angle CAB \cong$? [b] $\angle ACB \cong$? [c] $\angle CBA \cong$?

★**6.** Suppose the points R, S, T (not on one line) correspond respectively to the points W, X, Y so that $\overline{RS} \cong \overline{WX}$, $\overline{ST} \cong \overline{XY}$, $\overline{RT} \cong \overline{WY}$. Draw a picture. Which angles are congruent?

★**7.** Points A, B, C, D are congruent to E, F, G, H. Using your compass, draw the points E, F, G, H.



★**8.** Copy the triangle, using your compass and a straightedge.

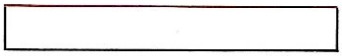


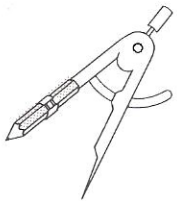
6-3 RULER AND COMPASS

A **straightedge** is a ruler without units marked on it. A **compass** is an instrument used to draw circles and parts of circles. Using only a straightedge (ruler) and compass, the ancient Greeks solved many geometric problems.

Drawings that can be made using only these tools are called **ruler-and-compass constructions**. (In making constructions, the ruler is used only to draw a line joining two points.) Copying an angle is one important example of ruler-and-compass construction.

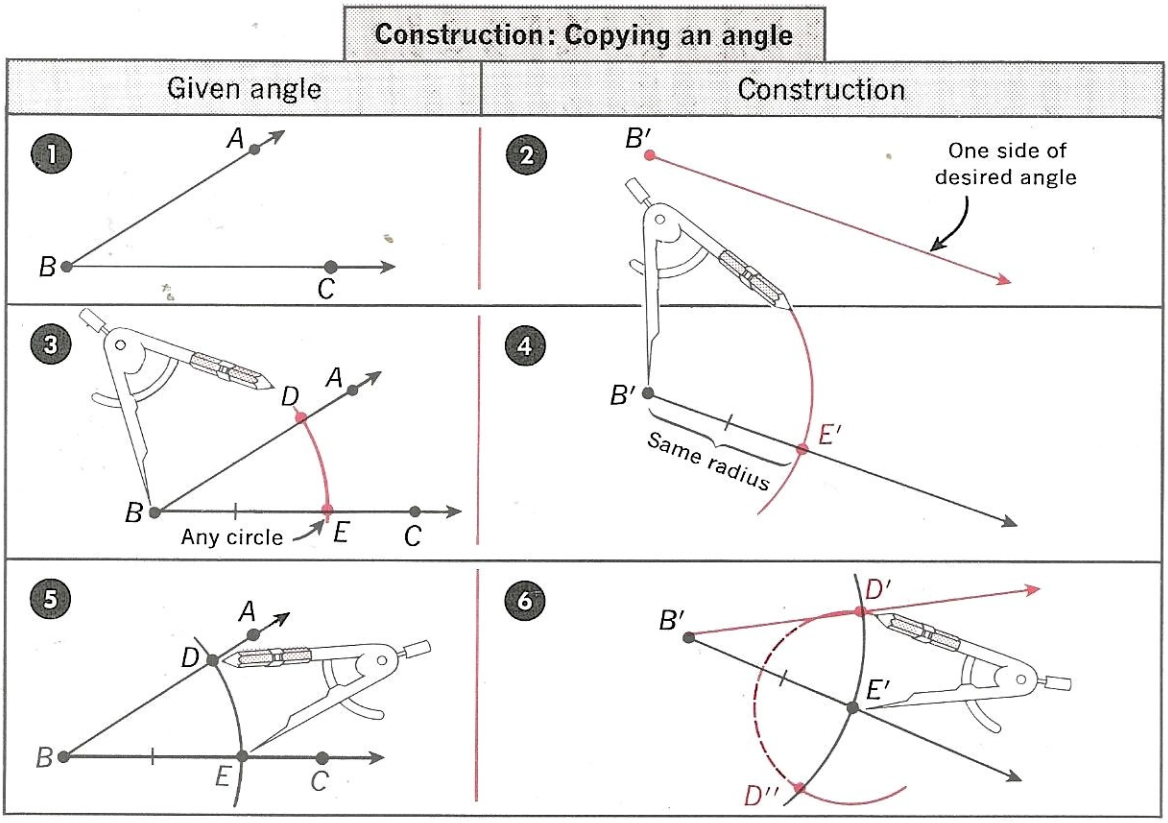
Straightedge





Compass

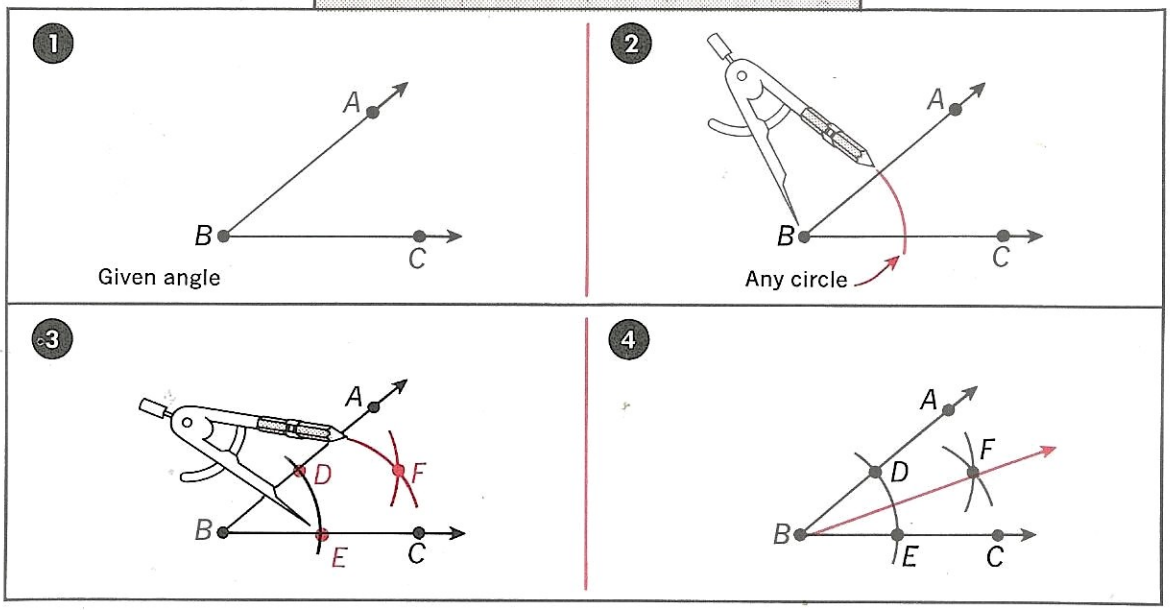
Construction: Copying an angle



Angle ABC is congruent to $\angle D'B'E'$. Is $\angle ABC$ also congruent to $\angle D''B'E'$? Can you tell why?

Another important ruler-and-compass construction is the bisecting of an angle, which is demonstrated in the following figures.

Construction: Bisecting an angle



We say that $\overrightarrow{BF}$ **bisects** $\angle ABC$ if $\angle ABF \cong \angle FBC$.

Discussion Exercises

1. In the construction for copying an angle, find

[**a**] a subset of points in step 6 that is congruent to the set $\{D, B, E\}$ in step 5

[**b**] another such subset of points in step 6

2. [**a**] In the construction for copying an angle, why is $\overline{BD} \cong \overline{B'D'}$?

[**b**] Why is the set $\{D, B, E\}$ congruent to the set $\{D', B', E'\}$?

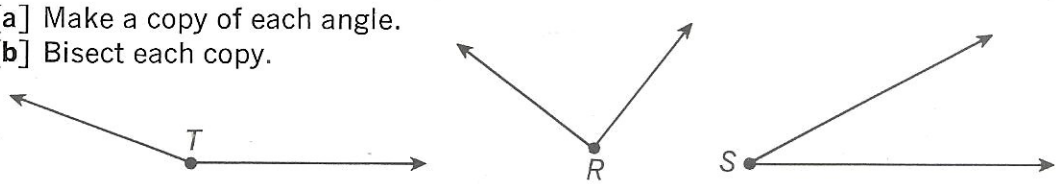
3. In the construction for bisecting an angle, how large must you make the circles with centers D and E?

Exercises

1. Using only a ruler and compass, draw three angles about the same size as angles R, S, and T.

[**a**] Make a copy of each angle.

[**b**] Bisect each copy.



2. Draw an angle M.
[a] Bisect it.
[b] Bisect each of the two new angles formed.

3. Ray AB is one side of $\angle BAC$. Ray AR is the bisector of $\angle BAC$. Show a construction that will give the other side ($\overrightarrow{AC}$) of $\angle BAC$.

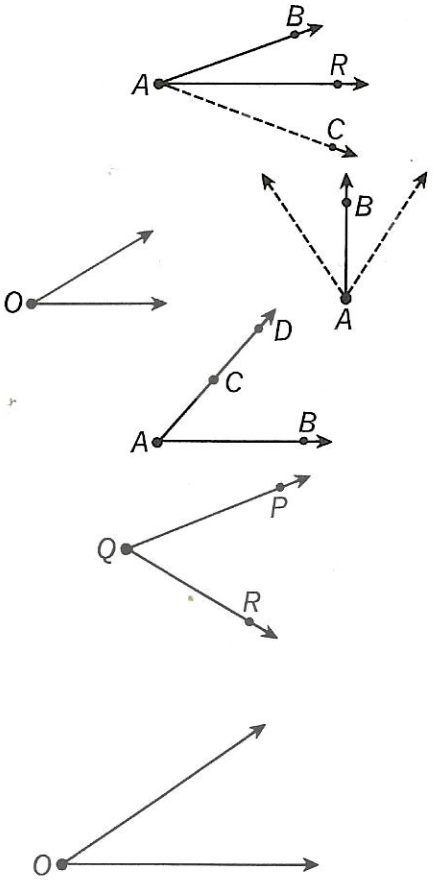
4. On your paper, draw a ray AB and an angle like $\angle O$. Using $\overrightarrow{AB}$ as a side for each, construct two angles, each congruent to $\angle O$.

5. Draw and label an angle like $\angle DAB$ shown on the right. Construct $\overrightarrow{CE}$ so that $\angle CAB \cong \angle DCE$.

6. Draw an angle STW about the size of $\angle PQR$. Construct an angle twice the "size" of $\angle STW$.

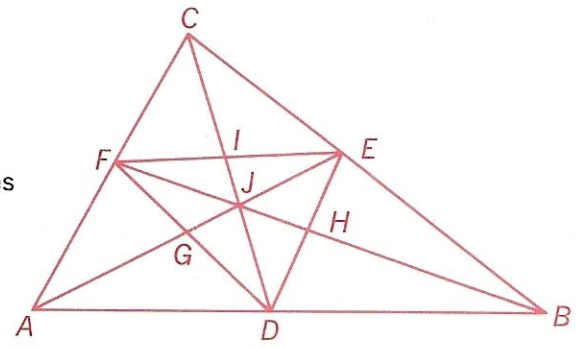
★ **7.** Draw $\angle XYZ$ about the size of $\angle PQR$. Construct an angle $1\frac{1}{2}$ times the "size" of $\angle XYZ$.

★ **8.** Draw a line and mark two points A and B on the line. Using ruler and compass, construct $\overrightarrow{AC}$ so that $\angle CAB \cong \angle O$. Now construct $\overrightarrow{AD}$ so that $\angle DAC \cong \angle O$. How many times can you copy $\angle O$ with vertex at A and yet stay on the same side of $\overleftrightarrow{AB}$?



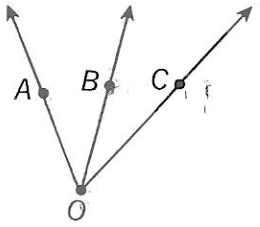
Think

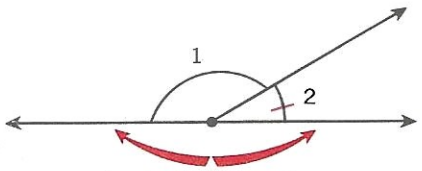
There are 47 different triangles in the figure. How many can you name?



6-4 RELATED ANGLES

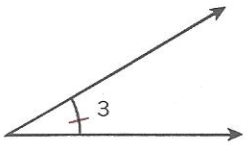
The figure shows a pair of **adjacent angles**; $\angle AOB$ is adjacent to $\angle BOC$. We also say that $\angle AOC$ is a **sum** of $\angle AOB$ and $\angle BOC$.





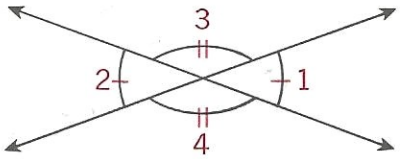
A straight angle

Angles 1 and 2 are **supplementary and adjacent**.

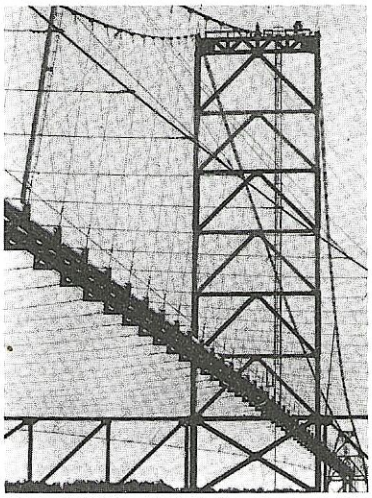


If $\angle 3$ is congruent to $\angle 2$, we say that $\angle 1$ and $\angle 3$ are **supplementary**. We sometimes say that $\angle 1$ is the supplement of $\angle 3$ or that $\angle 3$ is the supplement of $\angle 1$.

Some pairs of angles formed by intersecting lines are called **vertical angles**.



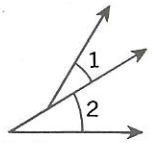
Angles 1 and 2 are vertical angles.
Angles 3 and 4 are vertical angles.



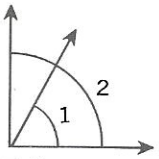
Can you see any adjacent or supplementary angles in this picture?

Discussion Exercises

1. In each exercise, explain why $\angle 1$ and $\angle 2$ are not adjacent angles.

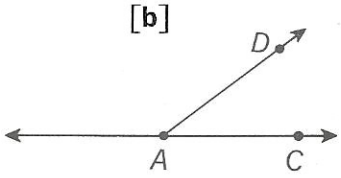


[a]

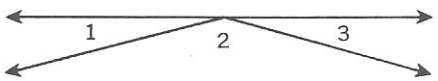


[b]

2. On which side of $\overleftrightarrow{AC}$ must $\overrightarrow{AB}$ be drawn in order to make $\angle BAC$ adjacent to $\angle DAC$?



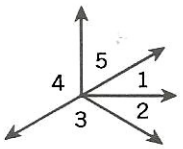
3. Tell why $\angle 1$ and $\angle 3$ are not vertical angles.



Exercises

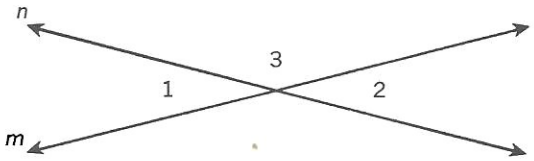
1. Using the figure to the right, name

[a] five pairs of adjacent angles
[b] two pairs of angles that are not adjacent



2. Draw and label a figure showing two adjacent angles, $\angle DEF$ and $\angle DEG$.

3. Lines m and n intersect and form a pair of vertical angles, $\angle 1$ and $\angle 2$. Copy and complete the following sentences.

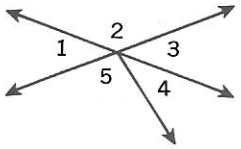


[a] Angle 3 is a supplement of $\angle 1$; $\angle 3$ is also a supplement of __?__.
[b] If two angles have the same angle for their supplement, the two angles are __?__.
[c] Since $\angle 3$ is a supplement of both $\angle 1$ and $\angle 2$, we know that $\angle 1$ and $\angle 2$ are __?__.
[d] Two vertical angles are __?__ to each other.

4. Copy the angles and draw supplements of each angle. Which is larger, the supplement of $\angle 1$ or the supplement of $\angle 2$?

5. Indicate which pairs of angles are

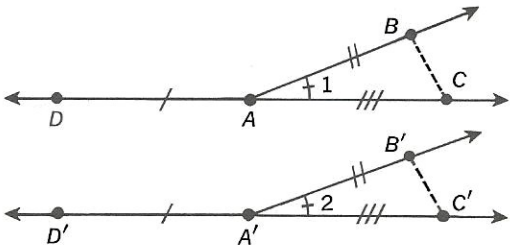
[a] vertical angles
[b] supplementary angles



6. Suppose you are given the following facts.

$$\angle 1 \cong \angle 2$$
$$\overline{AB} \cong \overline{A'B'}$$
$$\overline{AC} \cong \overline{A'C'}$$
$$\overline{AD} \cong \overline{A'D'}$$

List as many other congruence relations for the figure as you can.



Think

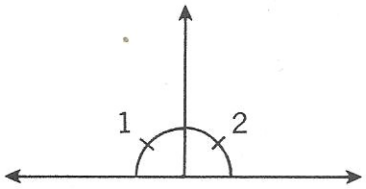
The figure compares the number of molecules of air in 1 cu in. of a good vacuum with the number of stars in the Milky Way. If there are about 1,000,000,000,000 stars in the Milky Way, about how many molecules are there in 1 cu in. of the best vacuum we can form?

Number of stars in Milky Way

Number of molecules in 1 cu in. of a good vacuum

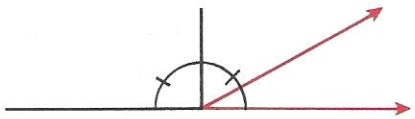
6-5 SPECIAL ANGLES

When two angles are congruent **and** supplementary, each angle is called a **right angle**. You might also say that a right angle is one that is congruent to its supplement.

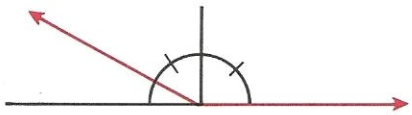


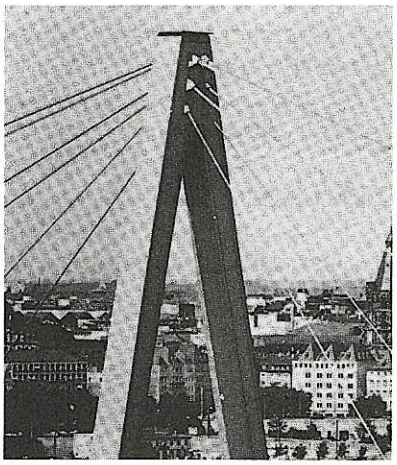
Angles 1 and 2 are right angles.

An angle smaller than a right angle is called an **acute angle**.



An angle larger than a right angle is called an **obtuse angle**.

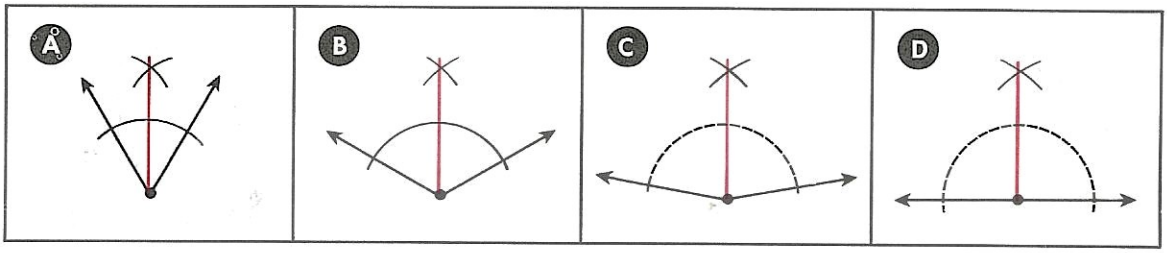




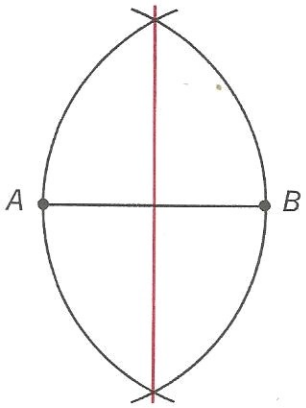
Can you see any acute or obtuse angles in this picture?

Discussion Exercises

1. Study each of the constructions.
   [a] In every case, what construction is being illustrated?
   [b] In which figure has a pair of right angles been constructed?
   [c] A pair of right angles can be constructed by starting with a straight angle and __?__.



2. The figure suggests a ruler-and-compass construction of a line that is **perpendicular** to a segment. The line also bisects the segment. This line is called the **perpendicular bisector** of the segment.
   [a] Explain how to complete this construction, starting with $\overline{AB}$.
   [b] What kind of angles are formed?



★3. Explain how you would start with a line m and point P on m and construct the perpendicular to m through P.

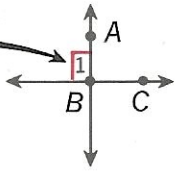
Exercises

1. Use **obtuse** or **acute** to fill each blank.
   [a] An angle that is congruent to an obtuse angle is an __?__ angle.
   [b] An angle that is supplementary to an acute angle is an __?__ angle.
   [c] If $\angle A$ and $\angle B$ are vertical angles with $\angle A$ acute, then $\angle B$ is __?__.
   [d] The supplement of an obtuse angle is an __?__ angle.
   [e] An angle that is less than a right angle is an __?__ angle.
   [f] If the sum of two angles is a right angle, then each angle is __?__.
   [g] A right angle is less than an __?__ angle.

2. Answer **true** or **false**.

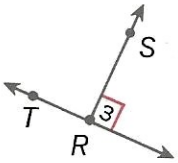
[a] Two right angles are congruent to each other.
[b] Any angle is congruent to its supplement.
[c] Two vertical angles are congruent to each other.
[d] An angle that is congruent to a right angle is itself a right angle.
[e] Any pair of adjacent angles are congruent to each other.
[f] If two angles are congruent, then their supplements are congruent.
[g] It is not possible for an obtuse angle to be congruent to an acute angle.
[h] Any two acute angles are congruent to each other.
[i] If the sum of two angles is a right angle, both angles are acute angles.
[j] If the sum of two angles is a straight angle, one of the angles must be obtuse.

3. When two lines meet to form a right angle, we say that these lines are **perpendicular** to each other. For example, $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{BC}$, and we write: $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. (Note: the box-type colored mark about $\angle 1$ indicates a right angle.)

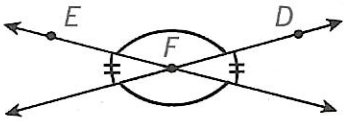


Tell which of the following figures show perpendicular lines. Then write statements, such as $\overleftrightarrow{AB} \perp \overleftrightarrow{XY}$, to show which lines are perpendicular.

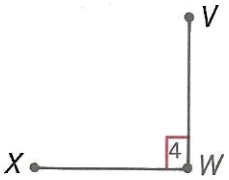
[a]



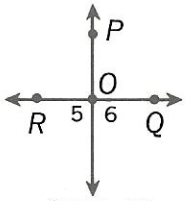
[b]



[c]

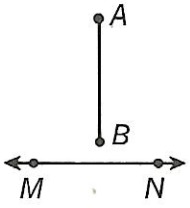


[d]

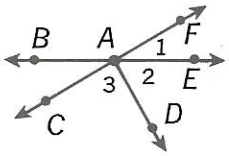


$\angle 5 \cong \angle 6$

[e]



[f]



The sum of $\angle 1$ and $\angle 2$ is a right angle, and $\overleftrightarrow{CAF}$ and $\overleftrightarrow{BAE}$ are straight lines.

4. Draw a line m and mark on line m three points A, B, and C, in that order. Now bisect $\angle ABC$. What kind of angles are formed?

5. [a] Draw a line segment and bisect it.
[b] Construct the perpendicular bisector of one of the smaller segments.

6. [a] Construct a right angle M.
[b] Bisect the right angle and label the smaller angles 1 and 2.
[c] Bisect $\angle 2$.
[d] Draw a supplement of $\angle 1$.

★ **7.** Construct a triangle that has one right angle and two sides congruent to $\overline{RS}$.

R ——— S

Think

Beginning with the numbers 12 and 20, build a set of numbers S. Include 12 and 20 as well as numbers that can be formed by additions and subtractions involving 12 and 20 and any other number in S.

$S = \{12, 20, 8, 32, \ldots\}$

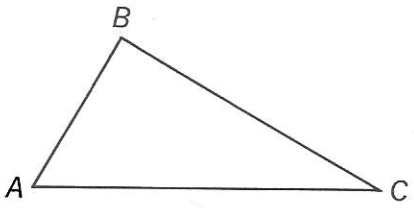
1. Find at least 10 more numbers in this set.

2. Describe the set of **all** such numbers.

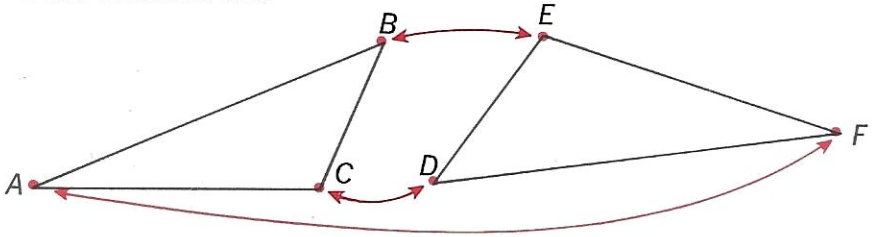
6-6 TRIANGLES

The parts of a triangle with **vertices** A, B, and C are

sides, $\overline{AB}$, $\overline{BC}$, $\overline{CA}$
angles, $\angle A$, $\angle B$, $\angle C$



If the vertices of two triangles are matched one to one, then the parts are matched too.



Then $\overline{BC}$ corresponds to $\overline{ED}$; $\angle A$ corresponds to $\angle F$, and so forth.

$\overline{BC} \longleftrightarrow \overline{ED}$ $\quad \angle A \longleftrightarrow \angle F$
$\overline{CA} \longleftrightarrow \overline{DF}$ $\quad \angle B \longleftrightarrow \angle E$
$\overline{AB} \longleftrightarrow \overline{FE}$ $\quad \angle C \longleftrightarrow \angle D$

4. Draw a line m and mark on line m three points A, B, and C, in that order. Now bisect $\angle ABC$. What kind of angles are formed?

5. [a] Draw a line segment and bisect it.
[b] Construct the perpendicular bisector of one of the smaller segments.

6. [a] Construct a right angle M.
[b] Bisect the right angle and label the smaller angles 1 and 2.
[c] Bisect $\angle 2$.
[d] Draw a supplement of $\angle 1$.

★ **7.** Construct a triangle that has one right angle and two sides congruent to $\overline{RS}$.

R ——— S

Think

Beginning with the numbers 12 and 20, build a set of numbers S. Include 12 and 20 as well as numbers that can be formed by additions and subtractions involving 12 and 20 and any other number in S.

$S = \{12, 20, 8, 32, \ldots\}$

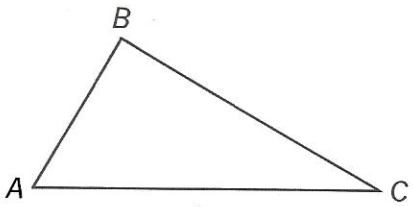
1. Find at least 10 more numbers in this set.

2. Describe the set of **all** such numbers.

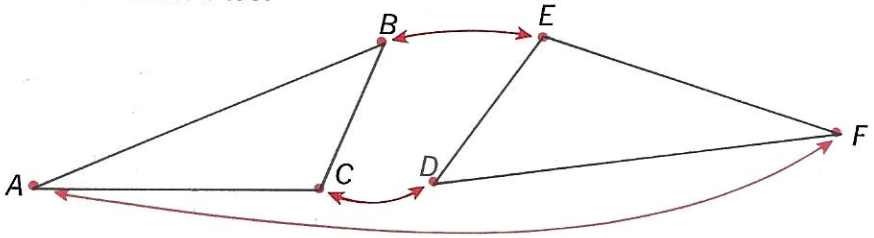
6-6 TRIANGLES

The parts of a triangle with **vertices** A, B, and C are

sides, $\overline{AB}$, $\overline{BC}$, $\overline{CA}$
angles, $\angle A$, $\angle B$, $\angle C$



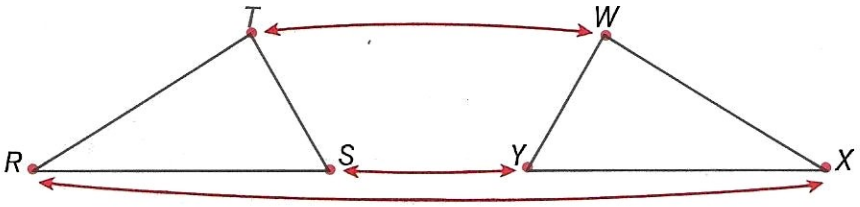
If the vertices of two triangles are matched one to one, then the parts are matched too.



Then $\overline{BC}$ corresponds to $\overline{ED}$; $\angle A$ corresponds to $\angle F$, and so forth.

$\overline{BC} \longleftrightarrow \overline{ED}$ $\quad \angle A \longleftrightarrow \angle F$
$\overline{CA} \longleftrightarrow \overline{DF}$ $\quad \angle B \longleftrightarrow \angle E$
$\overline{AB} \longleftrightarrow \overline{FE}$ $\quad \angle C \longleftrightarrow \angle D$

Suppose there is a correspondence between two triangles so that the corresponding parts are congruent as in the following example.



$\overline{RT} \cong \overline{XW}$ $\quad \angle S \cong \angle Y$
$\overline{TS} \cong \overline{WY}$ $\quad \angle R \cong \angle X$
$\overline{RS} \cong \overline{XY}$ $\quad \angle T \cong \angle W$

We write: $\triangle RST \cong \triangle XYW$.
We read: Triangle *RST* is congruent to triangle *XYW*.

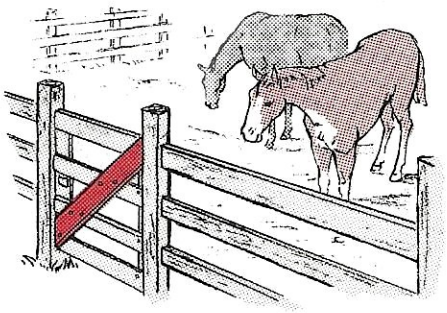
When we write the expression in the gray screen

$\triangle ABC \cong \triangle DEF$

we mean that the vertices correspond as indicated by the colored arrows and that each of six parts of one triangle is congruent to its corresponding part in the other.

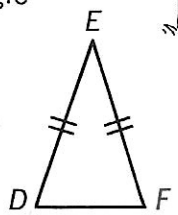
Discussion Exercises

1. $\triangle MNO \cong \triangle PQR$
- [a] What vertices correspond?
- [b] What sides correspond?
- [c] What angles correspond?

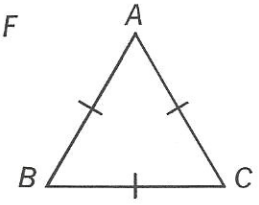


2. [a] Why is a crossbar put on a gate?
[b] Does the crossbar keep the angle at the corner of the gate from changing?

3. An **isosceles triangle** has two sides congruent. What two angles are then congruent?

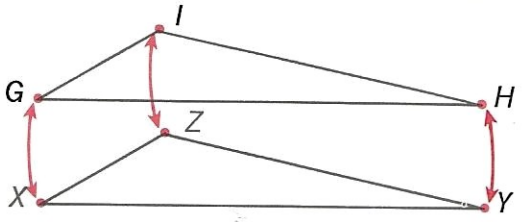


4. An **equilateral triangle** has all three sides congruent. Use Exercise 3 to explain why the three angles are congruent.



Exercises

1. $\triangle GHI \cong \triangle XYZ$. List the congruent angles and sides.

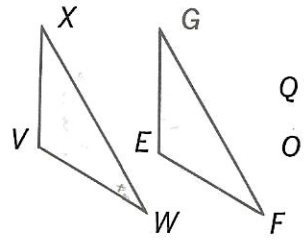


2. Two triangles, $\triangle ABC$ and $\triangle JKL$, are congruent, with the correspondence $A \leftrightarrow J$, $B \leftrightarrow K$, $C \leftrightarrow L$. Draw a picture and list the congruences.

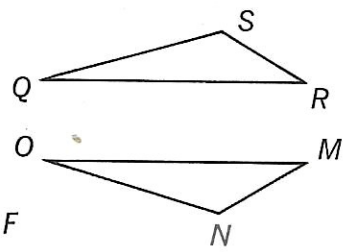
3. [a] If $\triangle DEF \cong \triangle GHI$, what vertices correspond?
[b] Draw a figure and list the congruences.

4. The triangles in each pair are congruent. List the congruences for each pair.

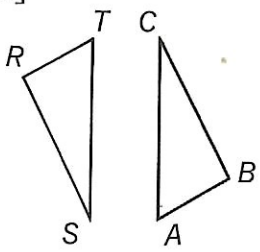
[a]



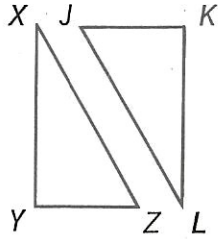
[b]



[c]

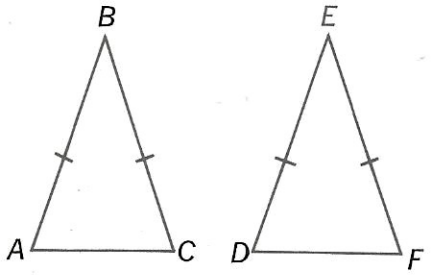


[d]

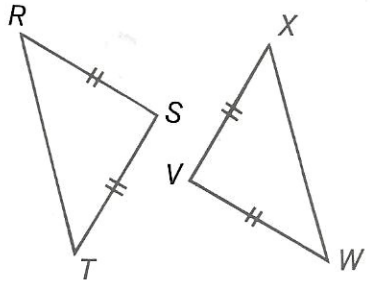


5. Write two different statements of congruence for each pair of isosceles triangles.

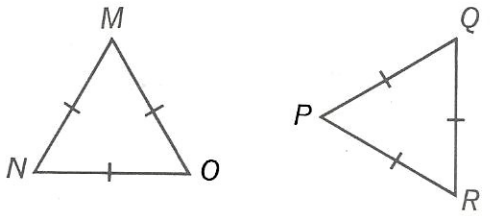
[a]



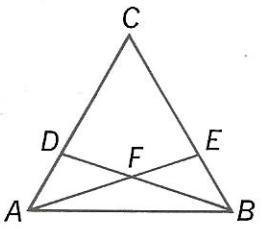
[b]



★**6.** How many congruences are there between the equilateral triangles? List them.

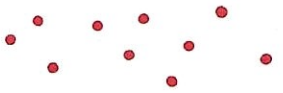


★**7.** In the figure, $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BE}$. Find three pairs of congruent triangles.



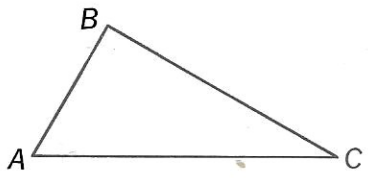
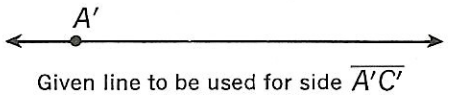
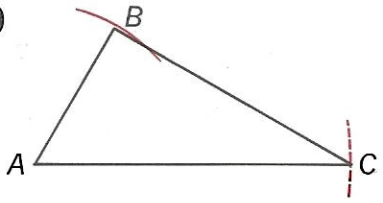
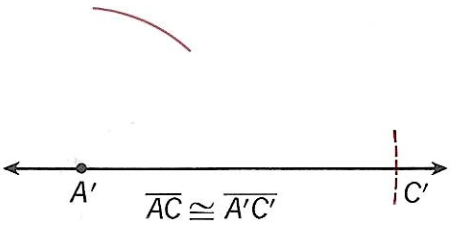
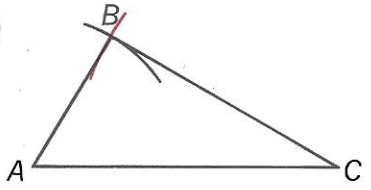
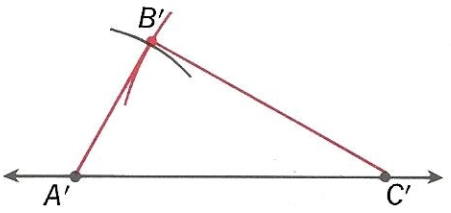
Think

Arrange ten points in five rows with four points in each row.



6-7 RULER AND COMPASS AGAIN

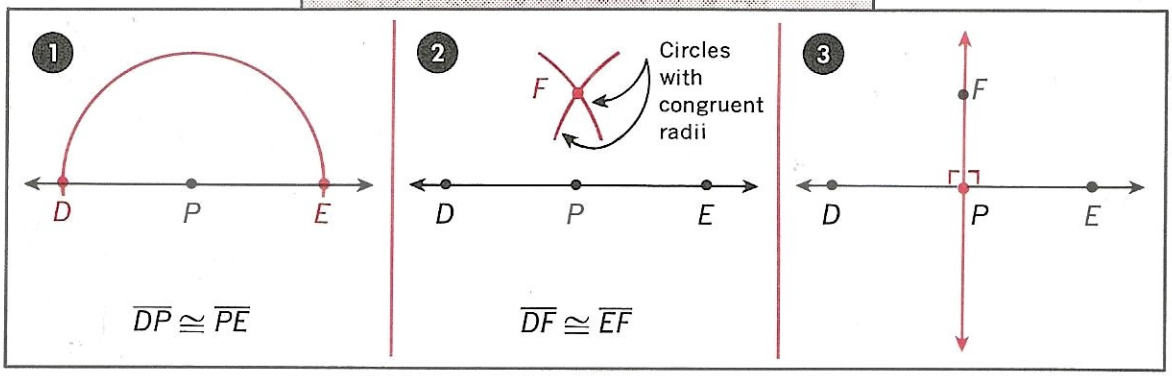
Construction: Copying a triangle

Given triangle	Construction
1 	2  Given line to be used for side $\overline{A'C'}$
3 	4  $\overline{AC} \cong \overline{A'C'}$
5 	6  $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$

Discussion Exercise A

In the diagram above, why is $\triangle ABC \cong \triangle A'B'C'$?

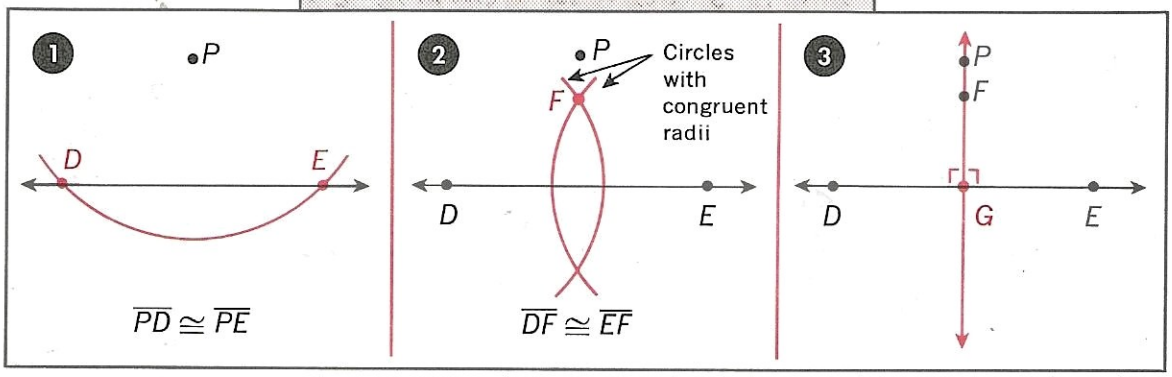
Construction: Drawing a perpendicular to a line through a point on the line



Discussion Exercises B

1. Why is $\angle DPF \cong \angle EPF$?
2. Why are $\angle DPF$ and $\angle EPF$ right angles?
3. ★ Why is $\triangle DPF \cong \triangle EPF$?

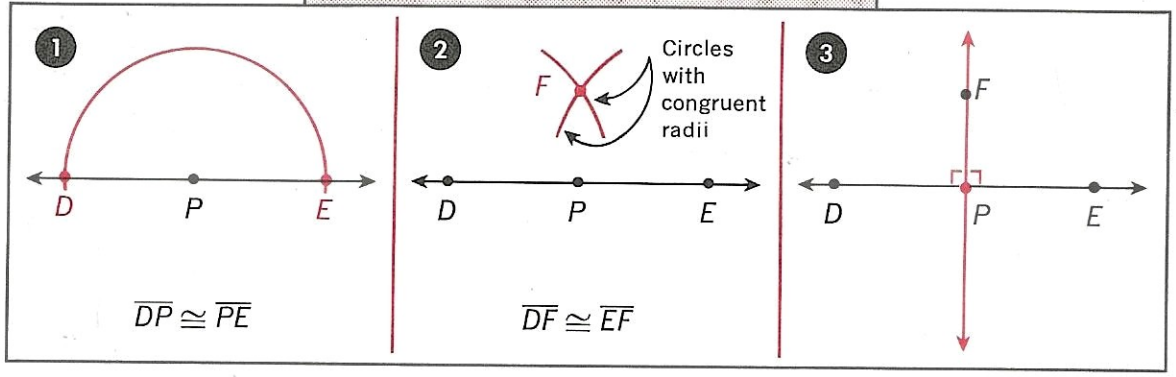
Construction: Drawing a perpendicular to a line from a point not on the line



★ Discussion Exercises C

1. Why is $\triangle DPF \cong \triangle EPF$ and why is $\angle DPG \cong \angle EPG$?
2. Why is $\overline{DG} \cong \overline{EG}$?
3. Why is $\angle DGP \cong \angle EGP$?
4. Why are $\angle DGP$ and $\angle EGP$ right angles?

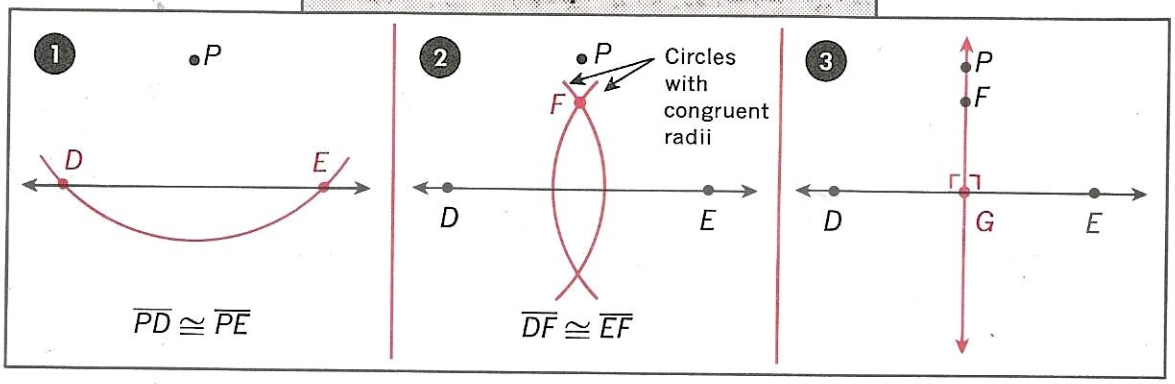
Construction: Drawing a perpendicular to a line through a point on the line



Discussion Exercises B

1. Why is $\angle DPF \cong \angle EPF$?
2. Why are $\angle DPF$ and $\angle EPF$ right angles?
3. ★ Why is $\triangle DPF \cong \triangle EPF$?

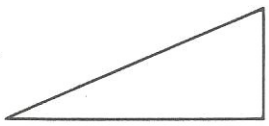
Construction: Drawing a perpendicular to a line from a point not on the line



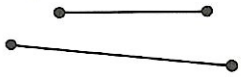
★ Discussion Exercises C

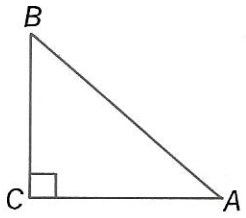
1. Why is $\triangle DPF \cong \triangle EPF$ and why is $\angle DPG \cong \angle EPG$?
2. Why is $\overline{DG} \cong \overline{EG}$?
3. Why is $\angle DGP \cong \angle EGP$?
4. Why are $\angle DGP$ and $\angle EGP$ right angles?

Exercises



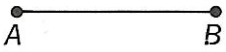
1. Draw a triangle ABC about twice the size of the one to the right. Then construct $\triangle PDQ$, such that $\triangle ABC \cong \triangle PDQ$.
2. Suppose $m \perp n$ at point B. Draw a picture.
3. Draw a line and call it m. Mark a point Q on it and construct $\overleftrightarrow{QX} \perp m$.
4. Draw a line n and mark a point A not on n. Construct a line through A that is perpendicular to n.
5. A **right triangle** is a triangle with one angle a right angle. Its **legs** are the two shorter sides. Construct a right triangle with legs congruent to these segments:



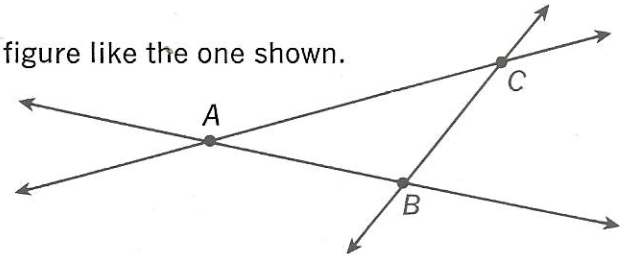


Right triangle ABC has legs $\overline{BC}$ and $\overline{AC}$.

6. Construct a right triangle with both legs congruent to $\overline{AB}$. Can you give a special name for this type of triangle?



7. Draw a figure like the one shown.



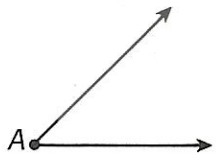
   [a] Construct a line p through B perpendicular to $\overleftrightarrow{AC}$. Call Q the point that line p and $\overleftrightarrow{AC}$ have in common. Segment BQ is called the **altitude** from B of $\triangle ABC$.

   [b] Construct a line r through C perpendicular to $\overleftrightarrow{AB}$. Let S be the point where r and $\overleftrightarrow{AB}$ intersect. Segment CS is called the altitude from C of $\triangle ABC$.

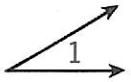
   [c] Construct a line t through A perpendicular to $\overleftrightarrow{BC}$. Let V be the point where t and $\overleftrightarrow{BC}$ intersect. Segment AV is called the altitude from A of $\triangle ABC$.

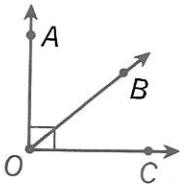
8. Draw a triangle and construct its altitudes.

★ **9.** Draw a ray from a point A'. Label this ray $\overrightarrow{A'B}$. Construct a supplement of $\angle A$ which has $\overrightarrow{A'B}$ as one side.

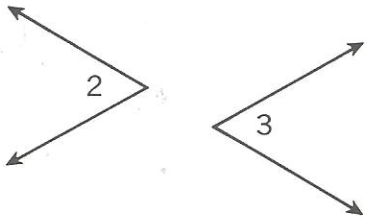


★**10.** If $\angle AOB$ and $\angle BOC$ are adjacent and acute angles and $\angle AOC$ is a right angle, then $\angle AOB$ and $\angle BOC$ are **complementary**. Construct an angle complementary to $\angle 1$.

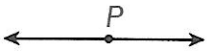




★**11.** Construct an angle which is a sum of $\angle 2$ and $\angle 3$.



Think

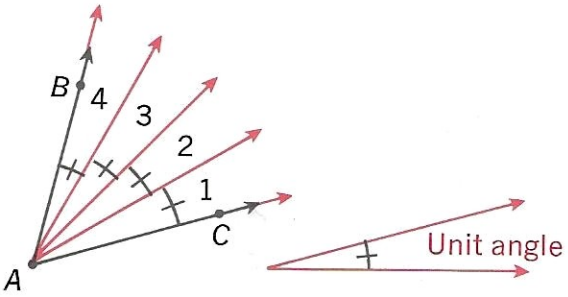


Pretend you have a "rusty" compass that has just one setting and cannot be changed. Use this rusty compass and your straightedge to construct the line that is perpendicular to a line at a point on the line.

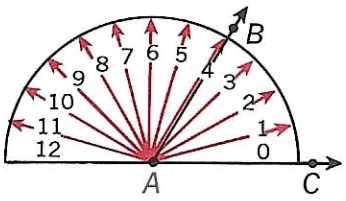
B. METRIC GEOMETRY

6-8 ANGLE MEASURE

To measure an angle, we must first choose a **unit angle**. Then we count unit angles as indicated in the figure. Angle A is the sum of 4 unit angles. We say that the measure of $\angle A$ is 4 units. We write this as $m\angle A = 4$ units.



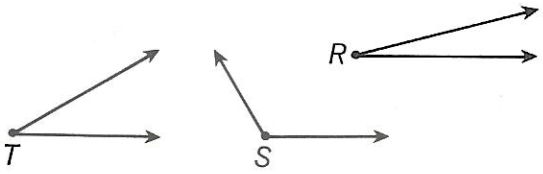
We can measure an angle by using a device that has a number of units already in place. Such a device is called a **protractor**.



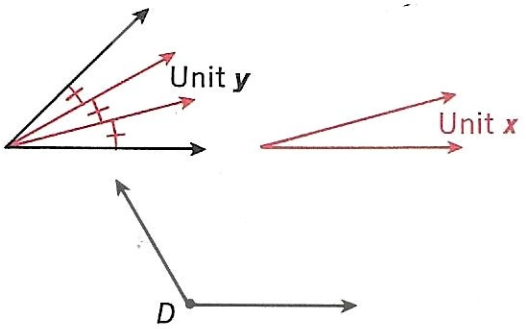
The measure of $\angle BAC$ is 4.

Discussion Exercises

1. Use $\angle R$ as a unit and estimate $m\angle S$ and $m\angle T$ to the nearest whole number of units.



2. Approximately what is $m\angle D$, using
[a] unit ***x***? [b] unit ***y***?

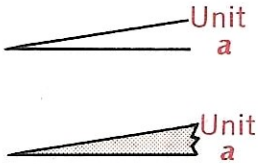


3. In Exercise 2, how would the measure of an angle, using unit ***x***, compare with the measure of the same angle when using unit ***y***?

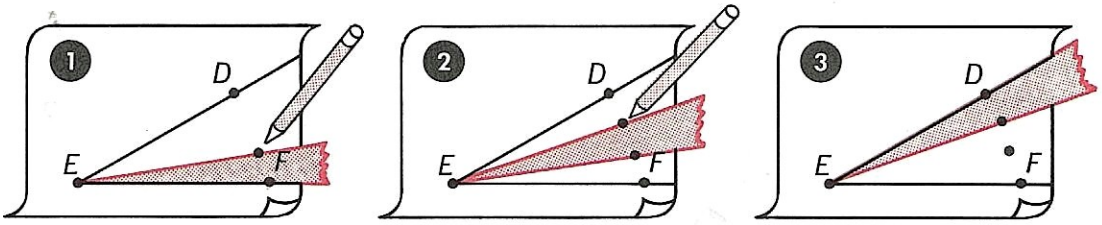
4. [a] Do you think a right angle would be a good angle unit?
[b] What is the measure of a straight angle, using a right angle as a unit?

Exercises

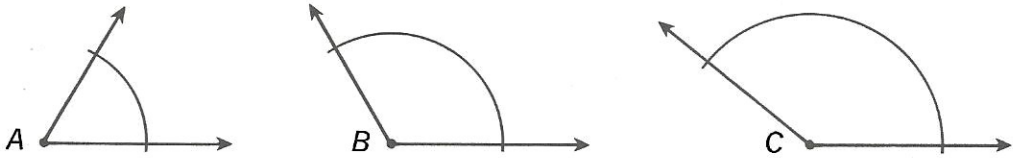
1. Copy the unit-angle ***a*** on a piece of paper and cut out a wedge of paper along the sides of your angle. Copy or trace $\angle A$, $\angle B$, and $\angle C$. Use the wedge to measure each angle as in the example.



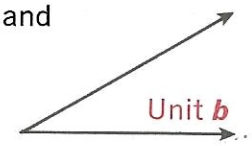
Example: Measure $\angle DEF$ by laying down the unit and marking a dot with your pencil, as shown in the three steps below.



The measure of $\angle DEF$ is 3 units.



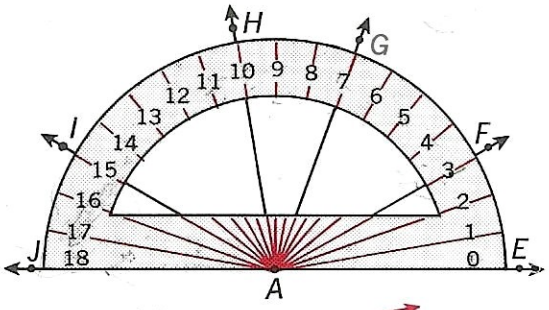
2. Copy the angles *A*, *B*, and *C* in Exercise 1. Using ruler and compass, "stack up" copies of the unit-angle ***b*** and so estimate their measures.



3. If you bisect the unit-angle ***b*** of Exercise 2 and use either of the smaller angles as a unit, what is $m\angle A$?

4. Using the given figure, find the measure of each angle.

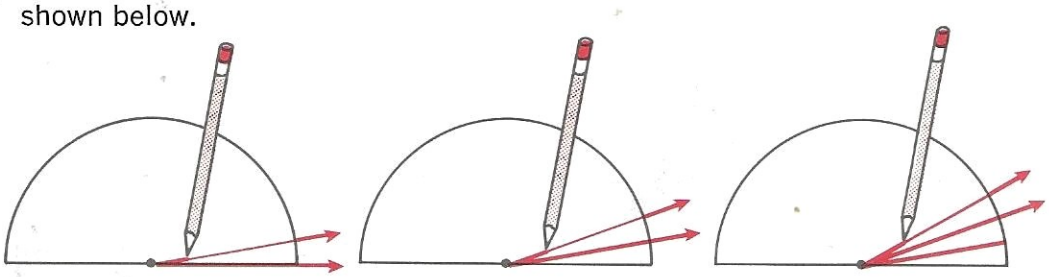
[a] $\angle EAF$	[e] $\angle EAJ$	[i] $\angle FAJ$
[b] $\angle GAE$	[f] $\angle JAI$	[j] $\angle GAH$
[c] $\angle EAH$	[g] $\angle HAJ$	[k] $\angle IAG$
[d] $\angle EAI$	[h] $\angle JAG$	[l] $\angle IAF$



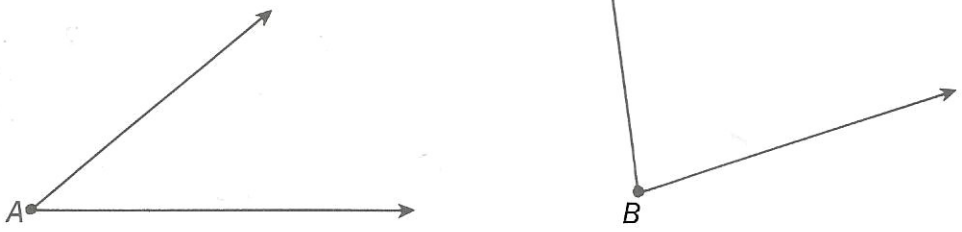
5. Here is how to make a protractor, using this unit.

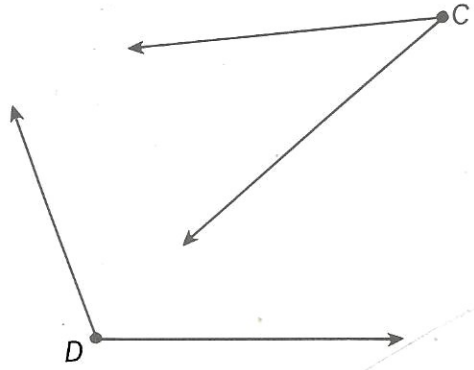
(A) On a piece of tracing paper, draw a circle with 4-cm radius.

(B) Cut out half of this circle and trace units side by side, as shown below.



Use the protractor you made to measure these angles to the nearest unit.





Think

The quotient of two consecutive even integers is $1\frac{1}{48}$. Find this pair of integers.

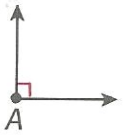
$a \div b = 1\frac{1}{48}$

6-9 DEGREES

The common practical unit for measuring angles is the **degree.** Here is an angle of 1 degree.

The size of a 1-degree angle is such that the measure of a right angle is 90 degrees.

We write: $m\angle A = 90°$

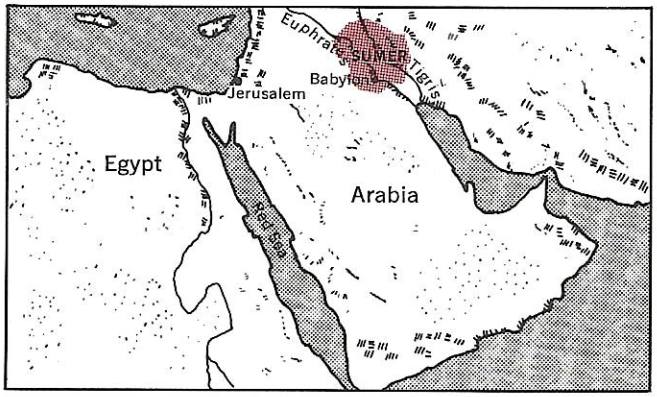


A straight angle has a measure of 180°, and in a full circle there are 360°.

Aztec calendar stone, showing signs for the months of the Aztec year, which consisted of 365 days with no leap year

We inherit the degree from the ancient Sumerians of 3000 B.C. The Sumerians used a lunar calendar of 12 months, each month having 30 days. (Every so often they had to add extra days to the year.)

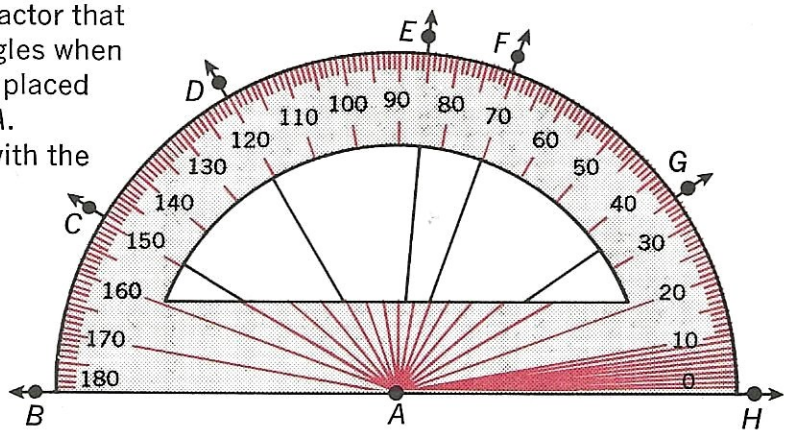
If the year were exactly 360 days, the earth in its annual passage around the sun would pass through 1° per day.



Sumer in 3000 B.C.
Capital: Babylon

The picture shows a protractor that can be used to measure angles when the degree is the unit. It is placed over some rays from point *A*.

Compare this protractor with the protractor in Exercise 4, Section 6–8.



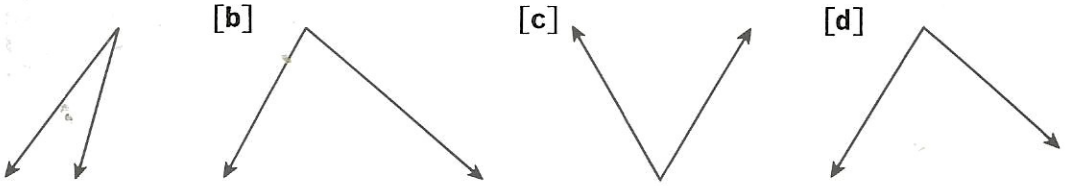
Exercises

1. Using the preceding picture of the protractor, give the measure of each angle in degrees.

[a] $\angle HAF$ [b] $\angle DAH$ [c] $\angle HAG$ [d] $\angle HAE$ [e] $\angle HAC$ [f] $\angle BAC$

2. With a protractor, find the measure of each angle.

[a] [b] [c] [d]



3. Using a protractor, draw angles with the following measures.

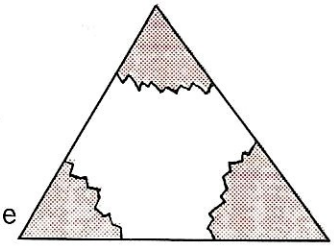
[a] 37° [b] 65° [c] 108° [d] 135°

4. Using your ruler and compass, construct a right angle and then measure it with your protractor.

5. What is the degree measure of a straight angle?

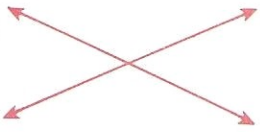
6. [a] Draw an angle. Copy it. Measure each angle.

[b] If two angles are congruent, what can you say about their measures?

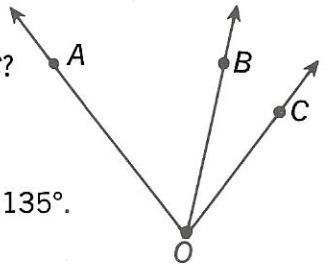


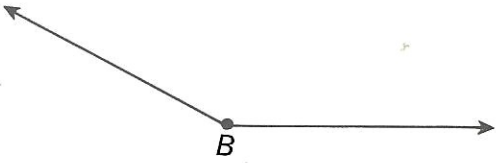
7. Draw a triangle and then cut out the triangle and tear off the three corners. Place the vertices together so as to form adjacent angles. What do you observe?

8. Draw two intersecting lines. Find a pair of vertical angles and measure each with your protractor. What can you say about measures of vertical angles?

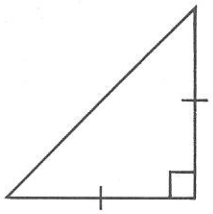


9. A degree is what part of a right angle?
10. A degree is what part of a straight angle?
11. Measure each angle. Does $m\angle AOC = m\angle AOB + m\angle BOC$?
12. Draw an angle and its supplement. Measure each angle, adding their measures. What do you observe?
13. Using your protractor, draw an angle of 45° and an angle of 135°. What can you say about these two angles?
14. Without measuring, why can you be sure that $m\angle B < 180°$?



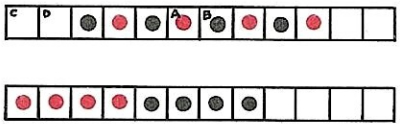


15. Angle R has measure 5 when a certain unit angle is used. This unit angle has a measure of 8°. What is $m\angle R$ in degrees?
16. Draw an equilateral triangle. Measure its angles. Add their measures.
17. Draw an isosceles right triangle. Measure its angles. Add their measures.
18. Draw any triangle. Measure its angles. Add their measures.



Think

Start with the first pattern and then move the coins (according to the rules) to get the second pattern. Let the red coins be pennies and the black coins be nickels.



The rules are:

- **A** A move consists of sliding a pair of adjacent coins from one position in the row to another. For example, you might first move penny **A** and nickel **B** to squares C and D.
- **B** The relative position of the coins in the pair that you move must remain the same. (In the move described in Rule A, penny **A** cannot go to square D, and nickel **B** cannot go to square C.)
- **C** Only four moves are allowed.

6-10 AREA

In Section 3–8, we counted unit squares to find area. It would be more convenient if we could find simpler methods for obtaining areas. To do this, we shall seek **formulas** for the areas of some common geometric regions.

Discussion Exercises

1. For a rectangle, it is easy to count squares.

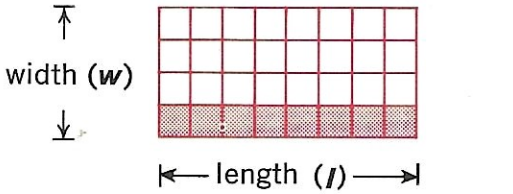
[a] How many unit segments along the bottom?

[b] How many unit squares in the bottom row?

[c] How many rows?

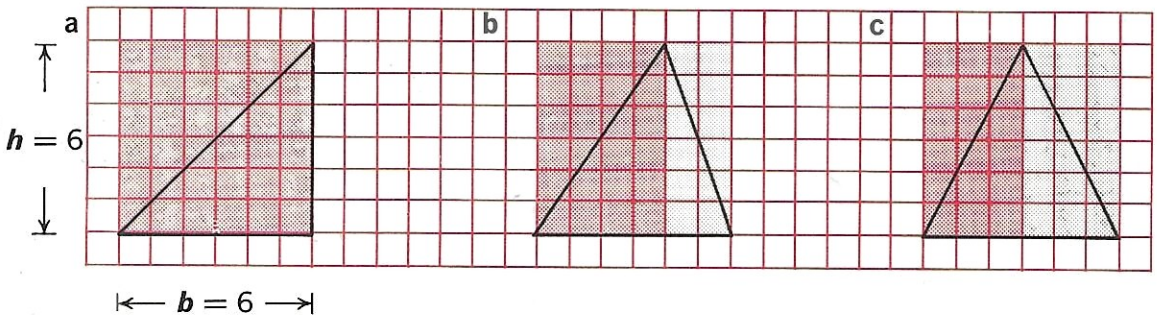
[d] How many total unit squares?

[e] Explain the formula for area of a rectangle: $A = w \cdot l$.

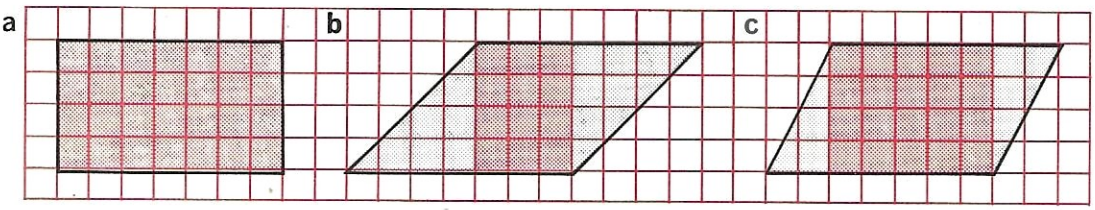


w rows of l squares
or l columns of w squares

2. For some triangles, it is also easy to count squares. Here are three triangles with the same base (b) and altitude (h). What is the area of each triangle?

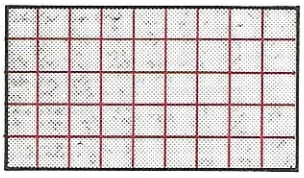


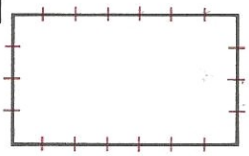
3. For some parallelograms, it is also easy to count squares. What is the area of each parallelogram?

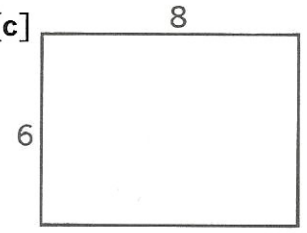


Exercises

1. Find the area of each rectangle.

[a] 

[b] 

[c] 

2. At 48¢ per square foot (sq ft), what does it cost to cover the floor of a kitchen that is 8 by 13 ft?

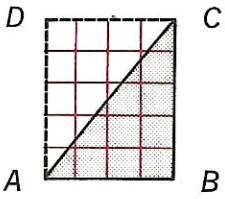
3. At 51¢ per sq ft, what does it cost to tile the floor and two adjacent walls of a bathroom that is 7 by 12 ft and 8 ft high?

4. [a] Is $\triangle ABC \cong \triangle CDA$?

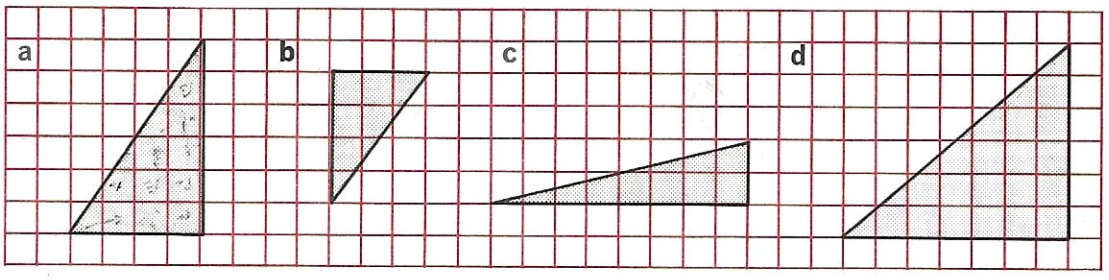
[b] What part of rectangle $ABCD$ is $\triangle ABC$?

[c] What is the area of the rectangle?

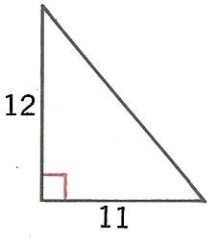
[d] What is the area of $\triangle ABC$?

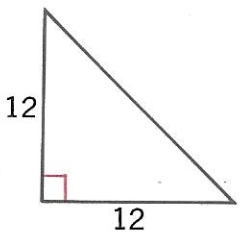


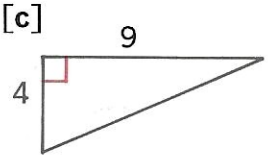
5. What is the area of each triangle?

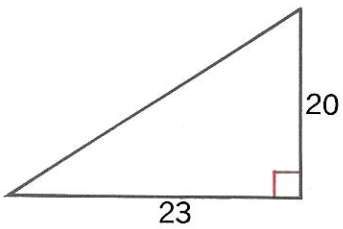


6. What is the area of each triangle? Dimensions are in inches.

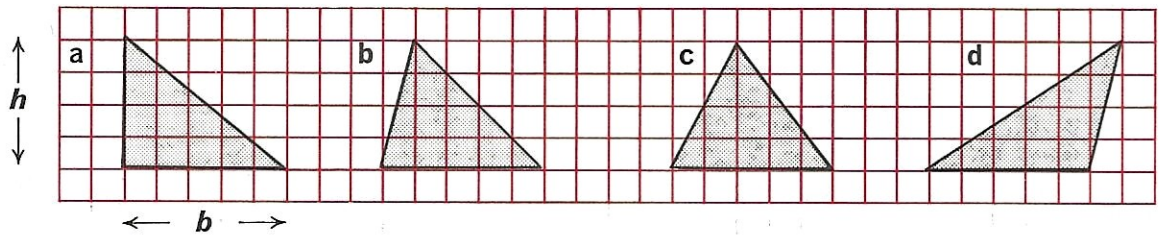
[a] 

[b] 

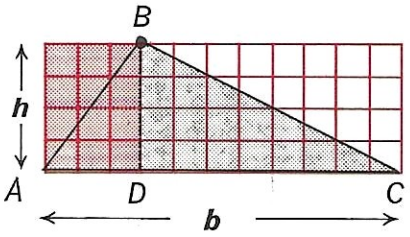
[c] 

[d] 

7. All four triangles have the same base and altitude. What is the area of each?



8. [a] What is the area of $\triangle ABD$?
[b] What is the area of $\triangle BDC$?
[c] What is the area of $\triangle ABC$?



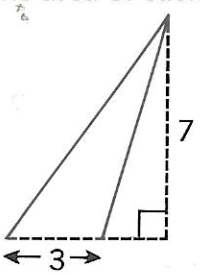
From this you should see that the formula for the area of a triangle is

$$A = \tfrac{1}{2} \cdot b \cdot h$$

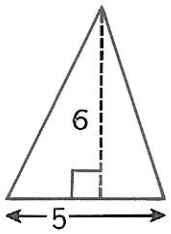
where b is the base and h is the altitude.

★**9.** Find the area of each triangle.

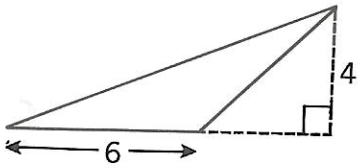
[a]



[b]

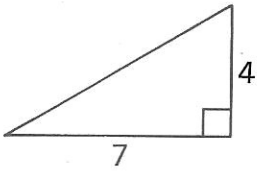


[c]



10. The dimensions of the triangles are in centimeters. Draw each triangle to actual size and find the area. Measure the other sides and give the perimeter to the nearest centimeter.

[a]



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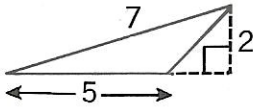

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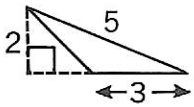
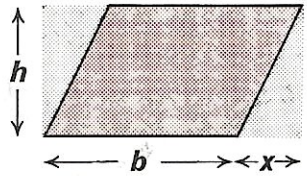


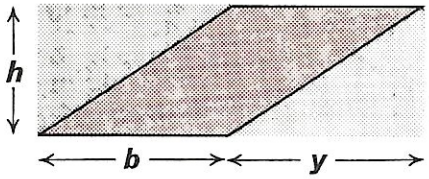
11. Find the areas of the triangles with the following dimensions.

[a] $b = 3$ ft, $h = 20$ in. [b] $b = 240$ cm, $h = 315$ cm

12. A triangle has an area of 216 sq in. Its base is 1 yd long. What is its height?

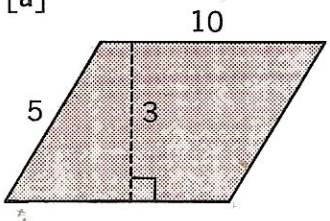
13. Both parallelograms have the same base and altitude. Give the areas
[a] of the large rectangles
[b] of the gray triangles
[c] of the parallelograms



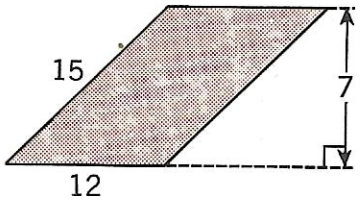


14. Find the area and perimeter of each parallelogram

[a]



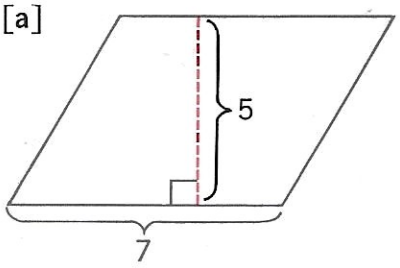
[b]



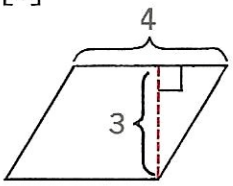
15. A triangle and a parallelogram have the same base. The areas of the two figures are equal. How do their heights compare?

16. Give the area of each parallelogram.

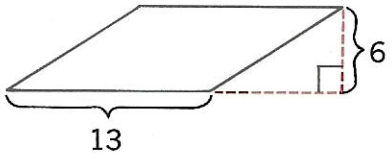
[a]



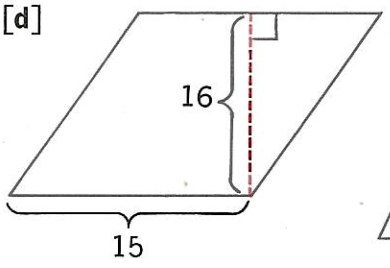
[b]



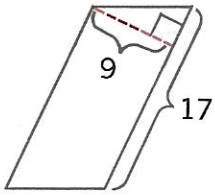
[c]



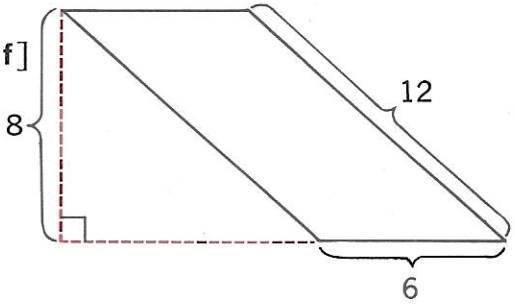
[d]



[e]

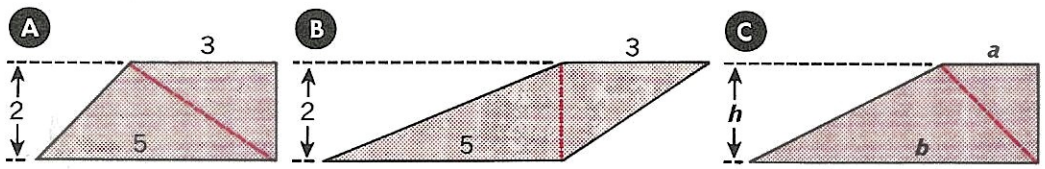


[f]



Think

What are the areas of these regions?

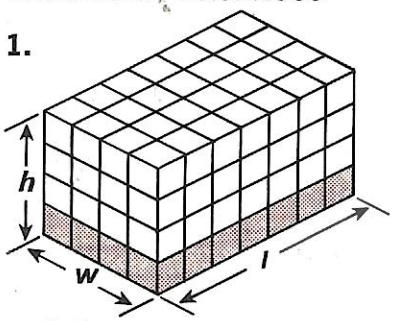


6-11 VOLUME

In Section 3–9, we counted unit cubes to find volume. It would be more convenient if we could find simpler methods for obtaining volumes. To do this we shall seek a formula for the volume of the region of space that we call a box.

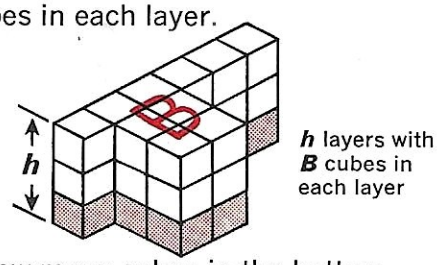
Discussion Exercises

1.



[a] How do you count the unit cubes in this rectangular box?
[b] How many squares in the base?
[c] How many unit cubes in the bottom layer?
[d] How many layers?
[e] How many cubes in all?
[f] Explain each formula for the volume of a rectangular box: Volume = area of base times altitude and $V = l \cdot w \cdot h$.

2. If a solid is formed of many layers of unit cubes, with the same number of cubes in each layer, then it is easy to count cubes. Suppose there are B cubes in each layer.

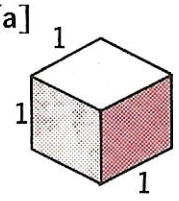


[a] How many cubes in the bottom layer?
[b] How many layers?
[c] How many cubes in all?
[d] Explain the formula $V = B \cdot h$.
[e] For the figure shown, what is the number B? The number h? What is the volume?

Exercises

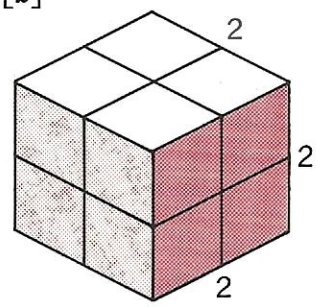
1. Give the volumes.

[a]



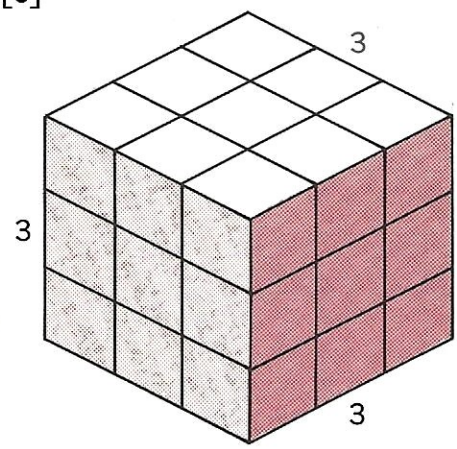
1-cm cube

[b]



2-cm cube

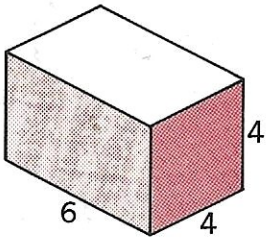
[c]



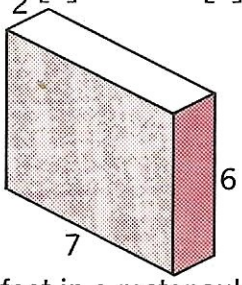
3-cm cube

2. Give the volumes.

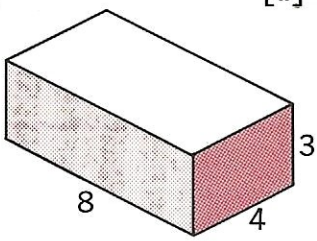
[a]



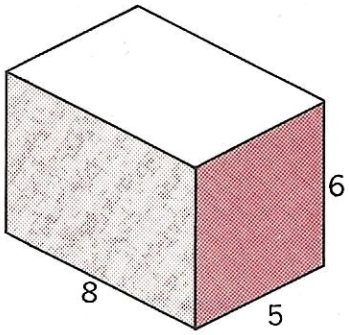
[b]



[c]



[d]



3. How many cubic feet in a rectangular box 2 by 9 by 12 ft?

4. One cubic foot contains $7\frac{1}{2}$ gallons (gal). How many gallons in a box 3 by 4 by 5 ft?

5. What is the volume of Figure A if

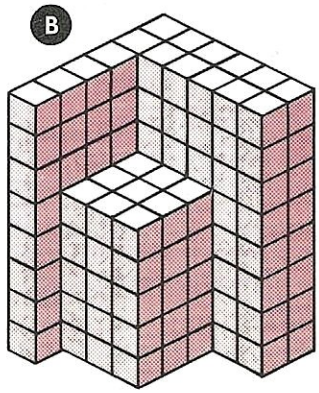
[a] all cubes are inch cubes?
[b] all cubes are 2-in. cubes?
[c] all cubes are 3-in. cubes?

6. What is the surface area of Figure A if

[a] all cubes are inch cubes?
[b] all cubes are 2-in. cubes?
[c] all cubes are 3-in. cubes?

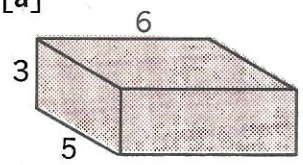
★**7.** Repeat Exercise 5, using Figure B.

★**8.** Repeat Exercise 6, using Figure B.

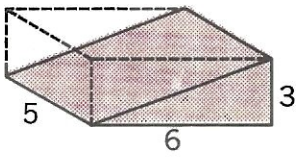


9. Give the volumes.

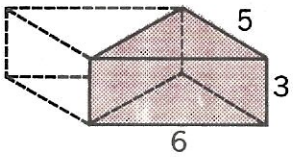
[a]



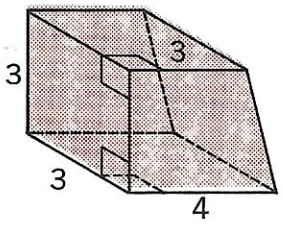
[b]



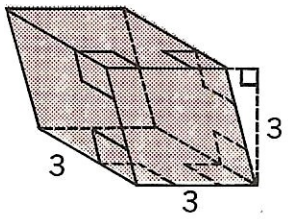
[c]



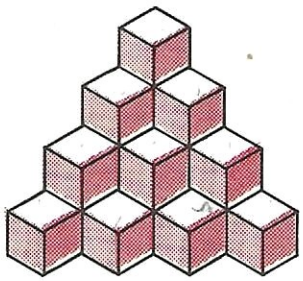
★[d]



★[e]



★**10.** Unit cubes are stacked in the corner of a room as in the picture.
[a] What is the volume?
[b] What is the surface area visible in the picture?



CHAPTER REVIEW

Key Concepts

1. Two pairs of points are congruent if they are equally far apart.

2. Two segments are congruent if their endpoints are equally far apart.

3. Any two rays are congruent to each other.

4. Any two lines are congruent to each other.

5. Two angles are congruent if any two points (one on each ray) are congruent to the two corresponding points on the other.

6. Two triangles are congruent if the vertices match so that the three corresponding segments and three corresponding angles are congruent.

7. When two angles are congruent **and** supplementary, each angle is a **right angle**.

8. A triangle with one right angle is a **right triangle**.

9. A ruler and compass can be used for
- copying an angle
- copying a triangle
- bisecting an angle
- constructing a line perpendicular to a given line at a point on the line and from a point not on the line
- constructing the perpendicular bisector of a segment

10. Angle measure involves counting unit angles.

11. A common unit for angle measure is the **degree**.

12. We may use formulas for finding the areas of

- rectangles: $A = w \cdot l$ where l = length, w = width
- triangles: $A = \frac{1}{2} \cdot b \cdot h$ where b = base, h = corresponding height (altitude)
- parallelograms: $A = b \cdot h$ where b = base, h = corresponding height (altitude)

13. The formula for the volume of a box is $V = w \cdot l \cdot h$ where $w \cdot l$ is the area of the base and h is the altitude to that base.

Vocabulary List

acute angle
adjacent angles
altitude
base
bisect
compass
complementary angles
congruent angles
congruent sets
congruent triangles
degree
equilateral triangle
isosceles triangle
obtuse angle
parts of a triangle:
  vertex
  side
  angle
perpendicular
perpendicular bisector
protractor
right angle
ruler
straightedge
supplementary angles
unit angle
vertical angles

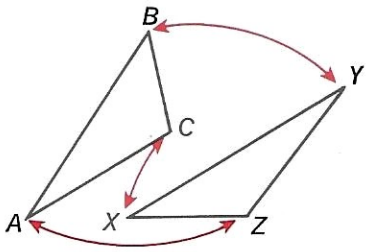
Review Exercises

1. Copy this segment with your ruler and compass. Mark a point on your segment that corresponds to C.

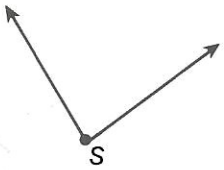
2. Draw a ray and mark a point Q on it. Copy this ray. Mark the point that corresponds to Q.

3. With this correspondence between the vertices of two triangles, what sides correspond?

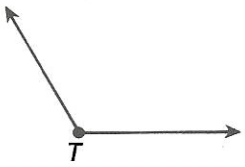
[a] $\overline{AB} \longleftrightarrow$?
[b] $\overline{BC} \longleftrightarrow$?
[c] $\overline{CA} \longleftrightarrow$?



4. Draw an angle about the size of $\angle S$ and copy it, using ruler and compass.



5. Draw an angle about the same size as $\angle T$ and bisect it.



6. Bisect a straight angle.

7. Draw the following figures.

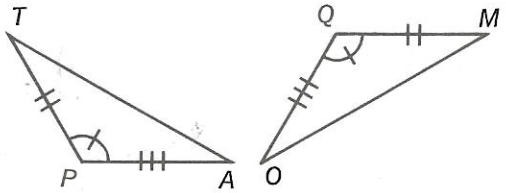
[a] Two adjacent angles
[b] An obtuse angle
[c] An acute angle and its complement
[d] An angle and its supplement

8. What is a right angle?

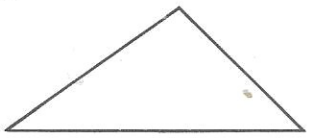
9. $\triangle RST \cong \triangle KLM$
[a] What vertices correspond?
[b] What angles are congruent?
[c] What sides are congruent?

10. $\triangle ABC \cong \triangle DEF$, and $\angle BCA$ is obtuse. What angle of $\triangle DEF$ is obtuse?

11. Angles and sides are congruent as shown. Why is $\overline{AT} \cong \overline{OM}$?



12. Draw a triangle about twice the size of the following one. Construct a copy of this triangle.

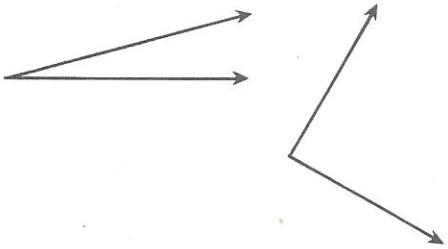


13. Draw a line n. Mark a point A on it. Using ruler and compass, draw a line m perpendicular to n at A.

14. Draw a line n. Mark a point A not on it. Using ruler and compass, draw a line m perpendicular to n through A.

15. What is a right triangle? Draw one. Label its vertices. What are the legs of your triangle?

16. With the smaller angle as the unit, estimate the measure of the larger angle.

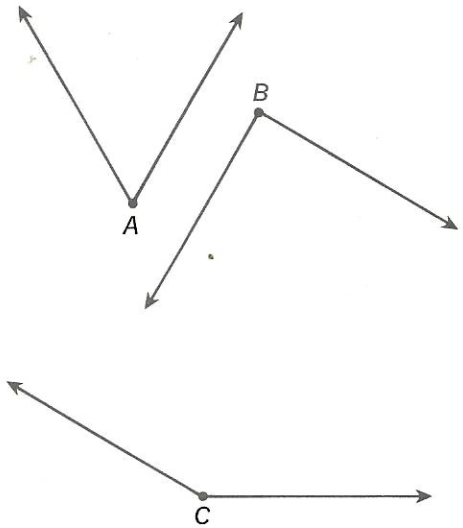


17. [a] With the right angle as a unit, what is the measure of a straight angle?
[b] How many right angles are there in a full circle?

18. What is the measure of a right angle in degrees?

19. Using your protractor, draw angles of
[a] 23° [c] 45° [e] 72° [g] 135°
[b] 37° [d] 60° [f] 105° [h] 150°

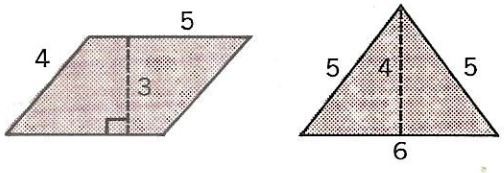
20. Measure these angles with your protractor.



21. What is the measure of the supplement of an angle measuring
[a] 38°? [b] 91°? [c] 147°?

22. Angles 1 and 2 are vertical angles, and $\angle 3$ is a supplement of $\angle 2$. If $m\angle 1 = 49°$, give the measure of
[a] $\angle 3$ [b] $\angle 2$

23. Give the area and perimeter of
[a] the parallelogram [b] the triangle



24. Explain how you count the number of cubes in a box with dimensions ***a***, ***b***, ***c***.

CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 23, 25, 62, 67–70.

BELL, E. T. *Men of Mathematics.* New York, Simon and Schuster, 1937; pp. 35–55.

JOHNSON, DONOVAN, and GLENN, WILLIAM. *Topology.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1960; entire pamphlet.

MIDONICK, HENRIETTA O. (ed.). *The Treasury of Mathematics.* New York, Philosophical Library, 1965; pp. 349–360.

NEELY, HENRY M. *Triangles.* New York, Crowell Company, 1962; pp. 1–53.

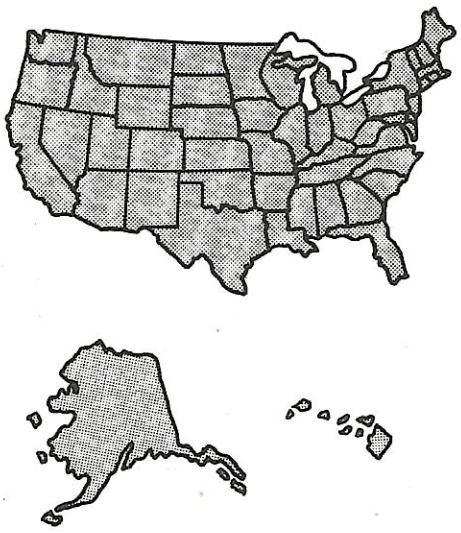
NORTON, M. SCOTT. *Geometric Constructions.* Exploring Mathematics on Your Own; St. Louis, Webster Publishing Company, 1963; pp. 1–33.

RAVIELLI, ANTHONY. *An Adventure in Geometry.* New York, Viking Press, 1957; pp. 42–57.

RUCHLIS, HY, and ENGELHARDT, JACK. *The Story of Mathematics—Geometry for the Young Scientist.* Irvington-on-Hudson, New York, Harvey House Publishers, 1958; pp. 1–33.

Activities

A. Find a small map of your state. Place a paper over the map and mark a set of 20 points congruent to a set of 20 points on the border of your map. (Choose the points carefully so you will have an outline of your state.) Now remove your paper from the map and mark 20 more "border" points. Finally, use your set of 40 points to help you draw, freehand, the outline of your state.



B. Find discussions of deductive and inductive reasoning. What is the difference? Give an example of each type of reasoning.

C. You may use constructions to illustrate the basic mathematical principles of commutativity and associativity for **addition.** See if you can demonstrate one of these constructions. (See *Geometric Constructions,* pp. 16–19.)

D. The commutative and associative principles for **multiplication** can be shown with constructions. Perform these constructions. Now try to show the distributive principle with constructions. (See *Geometric Constructions.*)

E. There is more than one way to construct a copy of a given triangle. Find discussions of constructions whose methods are different from those shown in this chapter (or, instead of reading about them, work them out on your own).

F. Figure out a new type of measure for angles. Show how to set up a protractor to use this new measure.

CUMULATIVE REVIEW

1. Solve the equations.
[a] $5 \cdot n = 75$ [c] $r \div 12 = 6$
[b] $23 - q = 17$ [d] $15 + (w \div 3) = 19$

2. Give the base ten numeral for $321_{(4)}$.

3. Study the tables and invent suitable function rules.

[a]

function rule	
n	$f(n)$
2	5
3	7
0	1
5	11
7	15

[b]

function rule	
n	$f(n)$
2	4
3	7
4	10
1	1
6	16

4. Find the two numbers a and b so that $a \cdot b = 45$ and $a \div b = 5$.

5. Write the regular numeral for each exercise.
[a] $(6 \cdot 10{,}000) + (5 \cdot 100) + (7 \cdot 10) + 1$
[b] $(3 \cdot 10^5) + (2 \cdot 10^4) + (8 \cdot 10^3) + (8 \cdot 10^2) + (5 \cdot 10) + 6$

6. Find the products.
[a] $37 \cdot 10$ [c] $80 \cdot 90$
[b] $523 \cdot 100$ [d] $650 \cdot 200$

7. Find the quotients and remainders.
[a] $8\overline{)3725}$ [b] $21\overline{)6574}$

8. Solve the equations.
[a] $3^3 \cdot 2^2 = n$ [c] $1232 \div 88 = r$
[b] $25 \cdot v = 175$ ★ [d] $4^a = 64$

9. A basketball team has played 20 games, winning 6 more than it lost. How many has it lost?

10. Bob is twice as old as Jerry. If Bob is 16 now, how old was Jerry five years ago?

11. Give the prime factorization for
[a] 30 [b] 54

12. If $N + P$ is an odd number and P is even, what can you say about N?

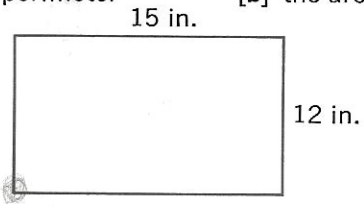
13. Find the LCM of
[a] 10 and 12 [b] 3, 8, and 12

14. List all the prime factors of
[a] 18 [b] 120

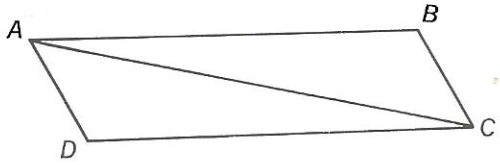
15. What is the GCF of each pair?
[a] 20 and 28 [b] 24 and 40

16. Pick out the correct word to complete each statement.
[a] If two segments have the same length, they are __?__.
(unequal, congruent, complementary)
[b] The area of a __?__ can be found by the formula $\frac{1}{2} \cdot b \cdot h$.
(square, rectangle, triangle)
[c] If the sum of two angles is a straight angle, the angles are __?__.
(supplementary, congruent, obtuse)
[d] A (an) __?__ angle has a measure less than 90°. (acute, obtuse, straight)
[e] Two rays with a common endpoint describe a (an) __?__.
(segment, half-plane, angle)

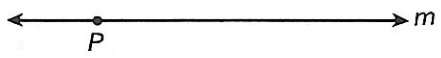
17. For the rectangle, find
[a] the perimeter [b] the area



18. If $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{AD}$, and $\overline{AC} \cong \overline{AC}$, write the statement which indicates that the two triangles are congruent.

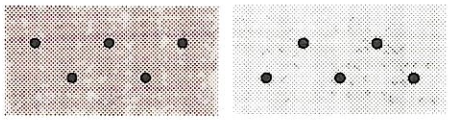


19. Pick a point P on line m. Using ruler and compass, bisect the straight angle at P. What can you say about the ray which bisects this angle?



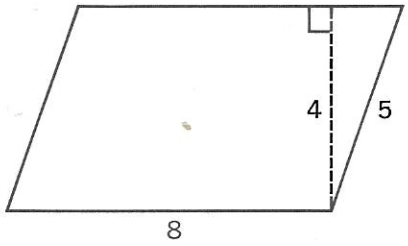
20. If $\overline{AB} > \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, what can you say about $\overline{AB}$ and $\overline{EF}$?

21. Are the two sets of points congruent?



22. For this parallelogram, find

[a] the perimeter **[b]** the area



23. If one angle is twice the measure of its supplement, what is its measure?

24. Solve the equations.

[a] $(20 \div n) \cdot 3 = 15$

[b] $7 \cdot (n + 4) = 49$

★**[c]** $(2 \cdot n) + (8 \cdot n) = 60$

★**[d]** $(n \div 6) \cdot (n \div 3) = 8$

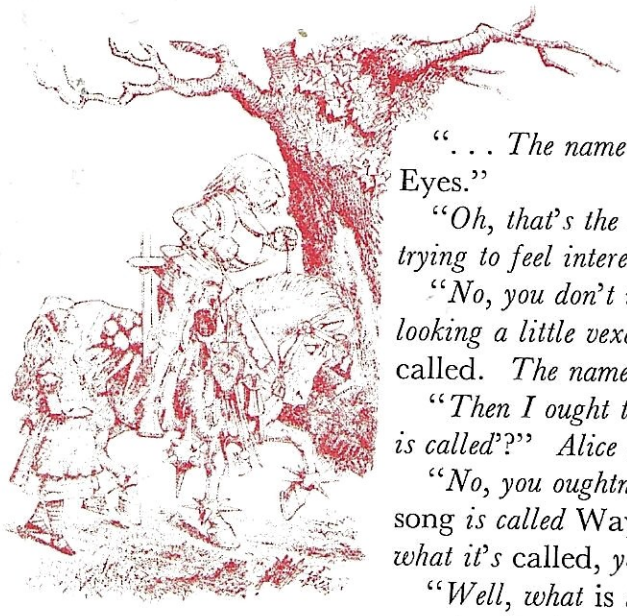
25. Estimate.

[a] $28 + 41 + 32$

[b] $159 - 62$

[c] $5998 + 3999 - 4001$

[d] $99 \cdot (348 + 51)$



". . . *The name of the song is called* Haddocks' Eyes."

"*Oh, that's the name of the song, is it?*" *Alice said, trying to feel interested.*

"*No, you don't understand,*" *the Knight said, looking a little vexed.* "*That's what the name is* called. *The name really* is The Aged Aged Man."

"*Then I ought to have said 'That's what the* song *is called'?*" *Alice corrected herself.*

"*No, you oughtn't: that's quite another thing!* The song *is called* Ways and Means: *but that's only what it's* called, *you know!*"

"*Well, what* is *the song, then?*" *said Alice, who was by this time completely bewildered.*

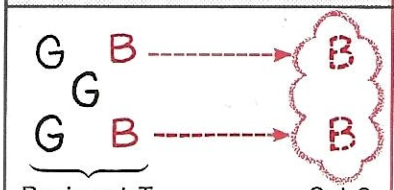
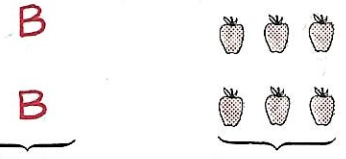
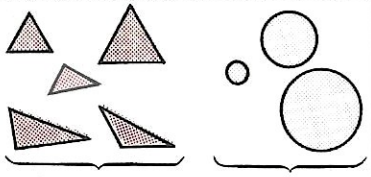
"*I was coming to that,*" *the Knight said.* "*The song really* is A-sitting On a Gate: *and the tune's my own invention.*"

7

Fractions and Rational Numbers

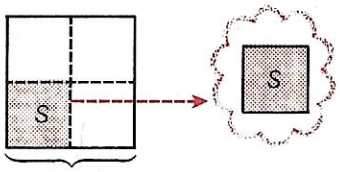
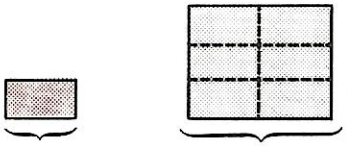
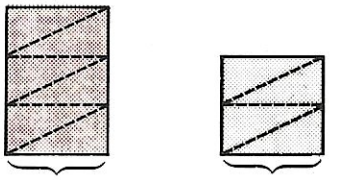
7-1 SETS, REGIONS, AND FRACTIONS

We use a fraction to compare a set S with a basic-set T. When we use fractions to compare sets, we are concerned with the **number of objects** in the sets, and not with their size.

Sets of objects		
S is a subset of T.	S is not a subset of T.	
 Basic-set T (students) — Set S (boys)	 Set S (boys) — Basic-set T (apples)	 Set S (triangles) — Basic-set T (circles)
We think: Set $S \rightarrow$ **2** boys. Basic-set $T \rightarrow$ **5** students. **We write:** $\frac{2}{5}$. This fraction compares the number of boys with the number of students. **We say: Two fifths** of the students are boys.	**We think:** Set $S \rightarrow$ **2** boys. Basic-set $T \rightarrow$ **6** apples. **We write:** $\frac{2}{6}$. This fraction compares the number of boys with the number of apples. **We say:** There are **two sixths** as many boys as there are apples.	**We think:** Set $S \rightarrow$ **5** triangles. Basic-set $T \rightarrow$ **3** circles. **We write:** $\frac{5}{3}$. This fraction compares the number of triangles with the number of circles. **We say:** There are **five thirds** as many triangles as there are circles.

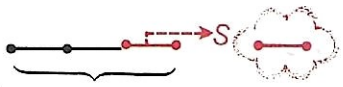
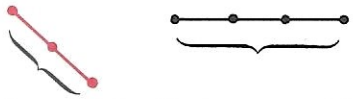
However, when we consider a fraction of a **region**, we are concerned with the **size** of each part **and** with the **number** of parts.

Sets of points (regions)

S is a subset of *T*.	*S* is not a subset of *T*.	
 Basic-region *T* Region *S*	 Region *S* Basic-region *T*	 Region *S* Basic-region *T*
We think: Region *S* → 1 part. Basic-region *T* → 4 parts.	**We think:** Region *S* → 1 part. Basic-region *T* → 6 parts.	**We think:** Region *S* → 6 parts. Basic-region *T* → 4 parts.
We write: $\frac{1}{4}$. This fraction compares region *S* with the basic-region *T*.	**We write:** $\frac{1}{6}$. This fraction compares region *S* with the basic-region *T*.	**We write:** $\frac{6}{4}$. This fraction compares region *S* with the basic-region *T*.
We say: One fourth of region *T* is shaded.	**We say:** Region *S* is **one sixth** as large as region *T*.	**We say:** Region *S* is **six fourths** as large as region *T*.

From the table above, you can see that when we divide a region into parts, such as sixths, we must have parts of the same size.

Sets of points (segments)

S is a subset of *T*.	*S* is not a subset of *T*.	
 Basic-segment *T* Segment *S*	 Segment *S* Basic-segment *T*	Segment *S* Basic-segment *T*
We think: Segment *S* → 1 part. Basic-segment *T* → 3 parts.	**We think:** Segment *S* → 2 parts. Basic-segment *T* → 3 parts.	**We think:** Segment *S* → 7 parts. Basic-segment *T* → 4 parts.
We write: $\frac{1}{3}$. This fraction compares the length of segment *S* with the length of basic-segment *T*.	**We write:** $\frac{2}{3}$. This fraction compares the length of segment *S* with the length of basic-segment *T*.	**We write:** $\frac{7}{4}$. This fraction compares the length of segment *S* with the length of basic-segment *T*.
We say: Segment *S* is **one third** of segment *T*.	**We say:** Segment *S* is **two thirds** as long as segment *T*.	**We say:** Segment *S* is **seven fourths** as long as segment *T*.

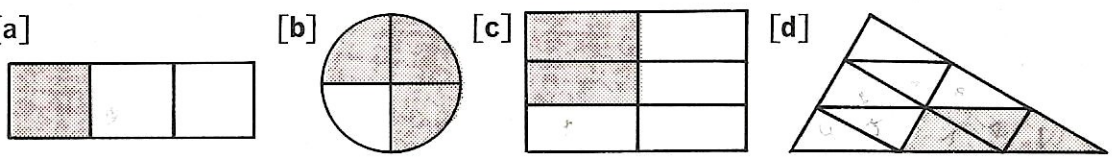
The numeral above the line of a fraction represents a number which is the **numerator** of the fraction. The numeral below the line represents a number which is the **denominator** of the fraction.

$\frac{2}{7}$ ← Numerator
$\frac{2}{7}$ ← Denominator

Exercises

1. For each figure, write a fraction that indicates the part of the region that is shaded.

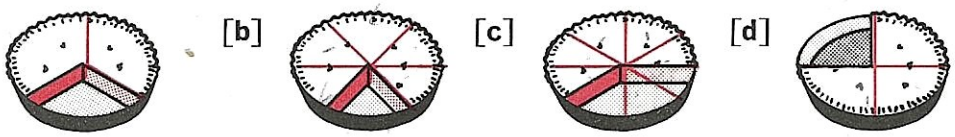
[a] [b] [c] [d]



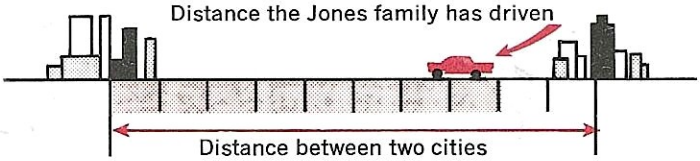
2. For each figure in Exercise 1, write a fraction that indicates the part of the region that is not shaded.

3. For each exercise, tell what part of the pie has been left. What part has been eaten?

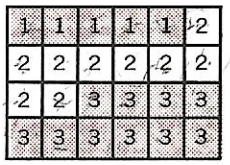
[a] [b] [c] [d]



4. According to the diagram, what part of the trip has the Jones family completed?

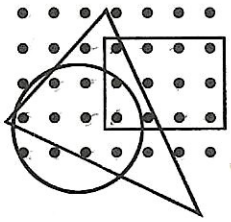


5. Tell what part of the set of squares is labeled with the number indicated.



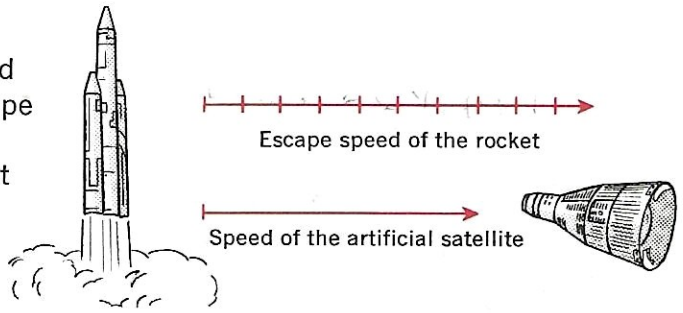
[a] 1 [c] 3 [e] 1 or 3
[b] 2 [d] 1 or 2 [f] 2 or 3

6. Tell what part of the set of dots is inside

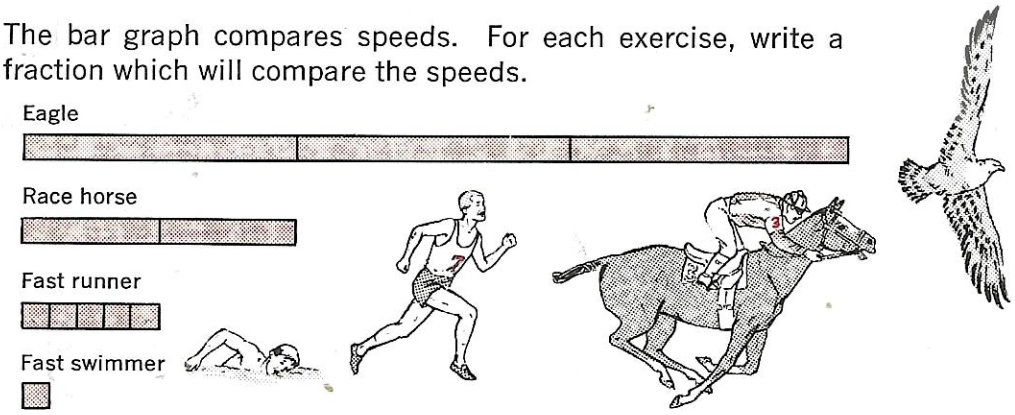


[a] the triangle
[b] the rectangle
[c] the circle
[d] both the rectangle and the circle

7. In the figure, the speed of an artificial satellite orbiting the earth is compared with the speed that a rocket must attain to escape from the earth and never return. The speed of the satellite is what part of the speed of the rocket?

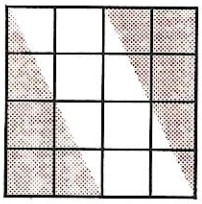


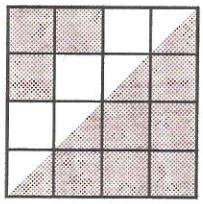
8. The bar graph compares speeds. For each exercise, write a fraction which will compare the speeds.

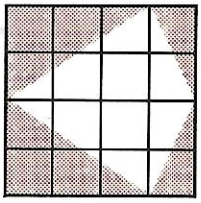


[a] Race horse to eagle
[b] Fast runner to race horse
[c] Fast swimmer to fast runner
[d] Fast runner to eagle
[e] Fast swimmer to race horse
[f] Fast swimmer to eagle

9. Write a fraction that gives the part of each region that is shaded.

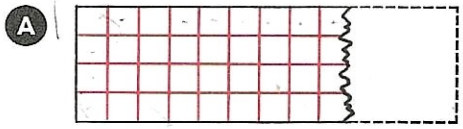
[a] 

[b] 

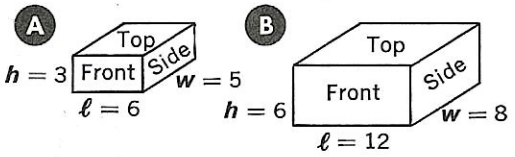
[c] 

10. The rectangles have the same length but different widths. Write a fraction that compares

[a] the area of A with the area of B
[b] the area of B with the area of A



11. Consider the two boxes.



Write the fraction that compares

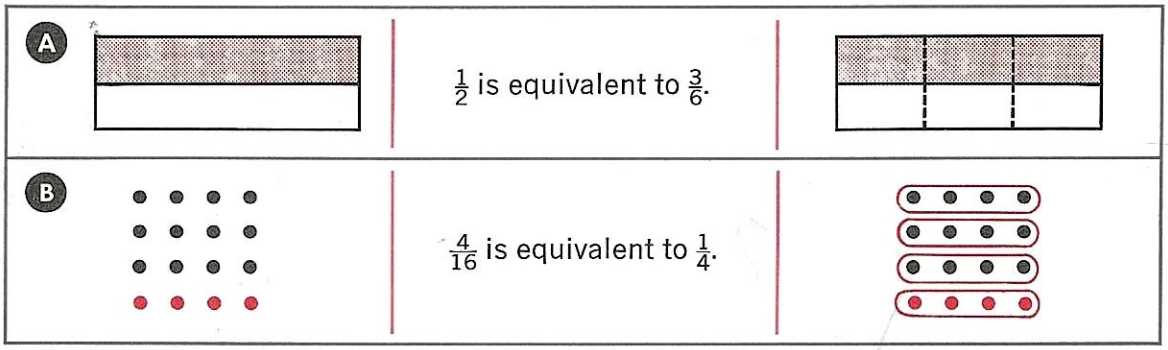
[a] the width of A to the width of B
[b] the length of A to the length of B
[c] the height of A to the height of B
[d] the area of the top of A to the area of the top of B
[e] the area of the front of A to the area of the front of B
[f] the volume of A to the volume of B
[g] the volume of B to the volume of A

Think

A microbe colony with plenty of food doubles its population every three hours. At noon the colony had a billion microbes. When was the population 500 million?

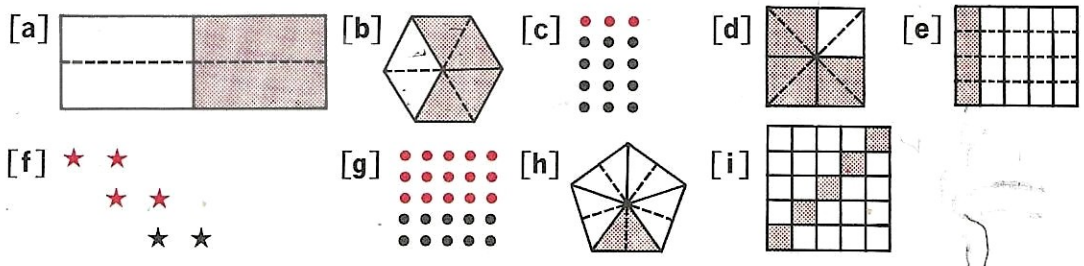
7-2 EQUIVALENT FRACTIONS

Two different fractions may be suggested by the same subset of a set or by the same part of a region. Two such fractions are **equivalent**. Examples A and B illustrate pairs of equivalent fractions.

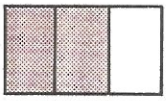
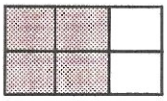
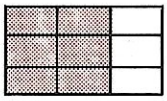
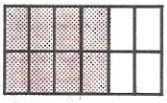


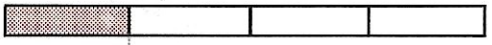
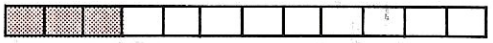
Exercises

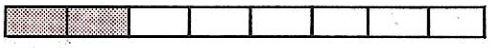
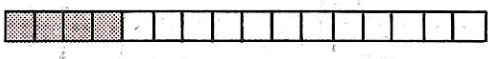
1. Write a pair of equivalent fractions suggested by each figure.



2. Write the fraction suggested by each figure.

[a]  [b]  [c]  [d] 

[e]  [g] 

[f]  [h] 

3. Write the next three fractions in each **set of equivalent fractions**.

[a] $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$ [b] $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \ldots\right\}$ [c] $\left\{\frac{3}{7}, \frac{6}{14}, \frac{9}{21}, \frac{12}{28}, \ldots\right\}$

4. Write the missing fractions for each row.

$\frac{2}{5}$	$\frac{2\cdot1}{5\cdot1}$	$\frac{2\cdot2}{5\cdot2}$	$\frac{2\cdot3}{5\cdot3}$	$\frac{2\cdot4}{5\cdot4}$	$\frac{2\cdot5}{5\cdot5}$	$\frac{2\cdot6}{5\cdot6}$
	↓	↓	↓	↓	↓	↓
	$\frac{2}{5}$	$\frac{4}{10}$	[a]	[b]	[c]	[d]
$\frac{5}{6}$	$\frac{5\cdot1}{6\cdot1}$	$\frac{5\cdot2}{6\cdot2}$	$\frac{5\cdot3}{6\cdot3}$	$\frac{5\cdot4}{6\cdot4}$	$\frac{5\cdot5}{6\cdot5}$	$\frac{5\cdot6}{6\cdot6}$
	↓	↓	↓	↓	↓	↓
	$\frac{5}{6}$	[e]	[f]	[g]	[h]	[i]
$\frac{4}{3}$	$\frac{4\cdot1}{3\cdot1}$	$\frac{4\cdot2}{3\cdot2}$	$\frac{4\cdot3}{3\cdot3}$	$\frac{4\cdot4}{3\cdot4}$	$\frac{4\cdot5}{3\cdot5}$	$\frac{4\cdot6}{3\cdot6}$
	↓	↓	↓	↓	↓	↓
	[j]	[k]	[l]	[m]	[n]	[o]

5. Copy each set of equivalent fractions and list three more equivalent fractions for each set.

[a] $\left\{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \ldots\right\}$ [c] $\left\{\frac{1}{5}, \frac{2}{10}, \ldots\right\}$ [e] $\left\{\frac{3}{2}, \ldots\right\}$ [g] $\left\{\frac{7}{10}, \ldots\right\}$

[b] $\left\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \ldots\right\}$ [d] $\left\{\frac{2}{7}, \frac{4}{14}, \ldots\right\}$ [f] $\left\{\frac{4}{9}, \ldots\right\}$ [h] $\left\{\frac{5}{3}, \ldots\right\}$

6. One of the fractions in each set is not equivalent to any of the other fractions in that set. Which fraction is it?

[a] $\left\{\frac{3}{4}, \frac{6}{8}, \frac{8}{12}, \frac{12}{16}, \frac{15}{20}\right\}$ [c] $\left\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{36}, \frac{15}{40}\right\}$ [e] $\left\{\frac{5}{4}, \frac{10}{6}, \frac{15}{12}, \frac{20}{16}, \frac{25}{20}\right\}$

[b] $\left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}\right\}$ [d] $\left\{\frac{2}{3}, \frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}\right\}$ [f] $\left\{\frac{6}{24}, \frac{5}{20}, \frac{2}{8}, \frac{4}{18}, \frac{3}{12}\right\}$

7. If n stands for any whole number except zero, what can you conclude about the pair of fractions?

$$\frac{7}{8} \text{ and } \frac{7 \cdot n}{8 \cdot n}$$

Think

Two dozen cookies and a loaf of bread cost 92 cents. Half a dozen cookies and two loaves of bread cost 58 cents. What is the cost of a loaf of bread?

7-3 REDUCING FRACTIONS TO LOWEST TERMS

A fraction is **in lowest terms** if the greatest common factor (GCF) of both numerator and denominator is 1. For each set of equivalent fractions, there is one fraction which is the lowest-terms fraction for that set. Finding the lowest-terms fraction for a given fraction is called **reducing to lowest terms.**

Fraction	Factors	GCF	Original fraction	Lowest-terms fraction
$\frac{12}{15}$	$\{1, 2, 3, 4, 6, 12\}$	3	$\frac{4 \cdot 3}{5 \cdot 3}$	$\frac{4}{5}$
	$\{1, 3, 5, 15\}$			
$\frac{9}{16}$	$\{1, 3, 9\}$	1	$\frac{9 \cdot 1}{16 \cdot 1}$	$\frac{9}{16}$
	$\{1, 2, 4, 8, 16\}$			
$\frac{24}{16}$	$\{1, 2, 3, 4, 6, 8, 12, 24\}$	8	$\frac{3 \cdot 8}{2 \cdot 8}$	$\frac{3}{2}$
	$\{1, 2, 4, 8, 16\}$			
$\frac{0}{6}$	$\{1, 2, 3, 4, 5, 6, 7, \ldots\}$	6	$\frac{0 \cdot 6}{1 \cdot 6}$	$\frac{0}{1}$
	$\{1, 2, 3, 6\}$			

Exercises

1. The factors of five numbers are given in the table. Using these factors, give the GCF of the numerator and denominator for each fraction.

12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
15	1, 3, 5, 15
16	1, 2, 4, 8, 16
18	1, 2, 3, 6, 9, 18

[a] $\frac{12}{14}$ [b] $\frac{14}{15}$ [c] $\frac{12}{16}$ [d] $\frac{12}{18}$ [e] $\frac{12}{15}$ [f] $\frac{14}{16}$ [g] $\frac{16}{16}$

2. Tell whether or not the fraction is in lowest terms.

[a] $\frac{10}{12}$ [b] $\frac{15}{24}$ [c] $\frac{8}{9}$ [d] $\frac{9}{14}$ [e] $\frac{0}{3}$ [f] $\frac{21}{27}$ [g] $\frac{25}{36}$ [h] $\frac{13}{39}$ [i] $\frac{11}{44}$ [j] $\frac{3 \cdot 7}{4 \cdot 7}$ [k] $\frac{2 \cdot 2}{3 \cdot 5}$ [l] $\frac{3 \cdot 5}{5 \cdot 7}$ [m] $\frac{7 \cdot 1}{9 \cdot 1}$ [n] $\frac{2 \cdot 5 \cdot 7}{3 \cdot 3 \cdot 11}$ [o] $\frac{4 \cdot 7}{5 \cdot 6 \cdot 11}$

3. Give a set of four fractions, each equivalent to the given lowest-terms fraction.

Example: $\frac{3}{8} \longrightarrow \left\{\frac{3}{8}, \frac{6}{16}, \frac{9}{24}, \frac{12}{32}\right\}$

[a] $\frac{4}{5}$ [b] $\frac{1}{7}$ [c] $\frac{5}{3}$ [d] $\frac{3}{10}$ [e] $\frac{2}{9}$ [f] $\frac{5}{6}$ [g] $\frac{1}{16}$

4. Find the lowest-terms fraction for each set.

[a] $\left\{\frac{4}{6}, \frac{6}{9}, \frac{10}{15}, \frac{20}{30}\right\}$

[b] $\left\{\frac{15}{60}, \frac{16}{64}, \frac{40}{160}, \frac{8}{32}\right\}$

[c] $\left\{\frac{90}{100}, \frac{900}{1000}, \frac{9000}{10,000}, \frac{90,000}{100,000}\right\}$

[d] $\left\{\frac{65}{52}, \frac{25}{20}, \frac{100}{80}, \frac{15}{12}\right\}$

5. If the prime factors of the numerator and denominator are known, then you can easily reduce a fraction. Note the examples.

(A) $\frac{3 \cdot 5}{7 \cdot 5}$ reduced to lowest terms is $\frac{3}{7}$.

(B) $\frac{2 \cdot 3 \cdot 7}{2 \cdot 3 \cdot 5}$ reduced to lowest terms is $\frac{7}{5}$.

Reduce the given fractions to lowest terms.

[a] $\frac{2 \cdot 3}{2 \cdot 5}$ [b] $\frac{2 \cdot 3}{3 \cdot 5}$ [c] $\frac{7 \cdot 11}{11 \cdot 13}$ [d] $\frac{19 \cdot 23}{19 \cdot 31}$ [e] $\frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 7}$ [f] $\frac{5 \cdot 7 \cdot 11}{2 \cdot 5 \cdot 7}$ [g] $\frac{7 \cdot 11 \cdot 19}{11 \cdot 13 \cdot 19}$ [h] $\frac{3 \cdot 3 \cdot 7}{3 \cdot 5 \cdot 7}$ [i] $\frac{11 \cdot 11 \cdot 13}{11 \cdot 13 \cdot 31}$ [j] $\frac{3 \cdot 5 \cdot 5 \cdot 7}{3 \cdot 5 \cdot 7 \cdot 13}$ [k] $\frac{2 \cdot 7 \cdot 11 \cdot 11}{7 \cdot 11 \cdot 13}$ [l] $\frac{3 \cdot 3 \cdot 19 \cdot 31}{7 \cdot 3 \cdot 19}$

6. Reduce the fractions to lowest terms. Not all the numerators and denominators are completely factored.

[a] $\frac{6 \cdot 7}{15 \cdot 7}$ [b] $\frac{9 \cdot 11}{3 \cdot 5 \cdot 11}$ [c] $\frac{10 \cdot 7 \cdot 13}{15 \cdot 13}$ [d] $\frac{5 \cdot 11 \cdot 13}{22 \cdot 13}$ [e] $\frac{39 \cdot 17}{15 \cdot 13}$ [f] $\frac{21 \cdot 15}{9 \cdot 7}$ [g] $\frac{33 \cdot 10 \cdot 6}{2 \cdot 30 \cdot 11}$ [h] $\frac{4 \cdot 14 \cdot 34}{21 \cdot 17 \cdot 16}$

7. To reduce fractions to lowest terms, you can think about "dividing out" common factors. Study the examples and then reduce each fraction to lowest terms.

A $\frac{36}{48} \longrightarrow \frac{36 \div 6}{48 \div 6} \longrightarrow \frac{6}{8} \longrightarrow \frac{6 \div 2}{8 \div 2} \longrightarrow \frac{3}{4}$

Notice that if you had divided both numerator and denominator by the GCF, you could have saved one step.

$$\frac{36}{48} \longrightarrow \frac{36 \div 12}{48 \div 12} \longrightarrow \frac{3}{4}$$

B $\frac{21}{27} \longrightarrow \frac{21 \div 3}{27 \div 3} \longrightarrow \frac{7}{9}$

[a] $\frac{12}{10}$ [b] $\frac{30}{42}$ [c] $\frac{63}{105}$ [d] $\frac{18}{81}$ [e] $\frac{50}{35}$ [f] $\frac{40}{36}$ [g] $\frac{175}{105}$ [h] $\frac{98}{77}$

8. Give the missing numerator or denominator in order to form fractions equivalent to the given lowest-terms fraction.

[a] $\frac{3}{4} \longrightarrow \left\{\frac{\square}{20}, \frac{9}{\square}, \frac{\square}{40}, \frac{15}{\square}\right\}$ [c] $\frac{3}{2} \longrightarrow \left\{\frac{\square}{8}, \frac{21}{\square}, \frac{6}{\square}, \frac{18}{\square}\right\}$ [e] $\frac{5}{6} \longrightarrow \left\{\frac{\square}{18}, \frac{\square}{72}, \frac{40}{\square}, \frac{75}{\square}\right\}$

[b] $\frac{2}{7} \longrightarrow \left\{\frac{\square}{21}, \frac{14}{\square}, \frac{6}{\square}, \frac{\square}{70}\right\}$ [d] $\frac{8}{9} \longrightarrow \left\{\frac{24}{\square}, \frac{\square}{45}, \frac{72}{\square}, \frac{\square}{72}\right\}$ [f] $\frac{7}{11} \longrightarrow \left\{\frac{\square}{22}, \frac{77}{\square}, \frac{\square}{77}, \frac{91}{\square}\right\}$

Think

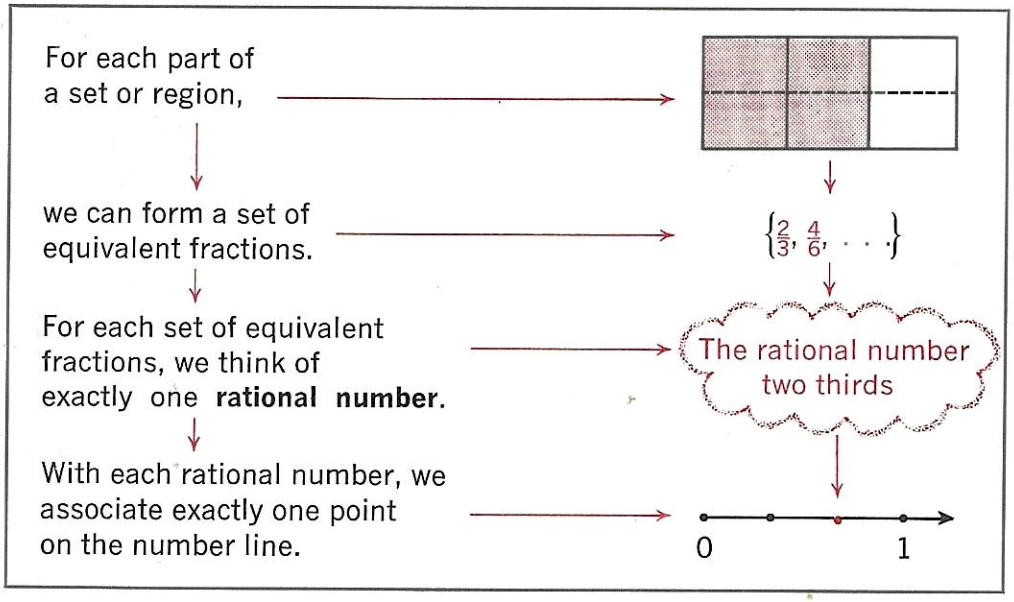
Discounting wind resistance and starting from rest, an object will fall approximately as indicated in the table.

$16 \cdot 1^2$ ft in 1 sec
$16 \cdot 2^2$ ft in 2 sec
$16 \cdot 3^2$ ft in 3 sec
$16 \cdot n^2$ ft in n sec

If a skydiver jumps from 10,000 ft, about how many seconds should he wait in order to pull his ripcord at about 3600 ft?

Photographed just before his parachute opened, this skydiver wears special clothing in order to make it easier for observers to follow his descent.

7-4 RATIONAL NUMBERS

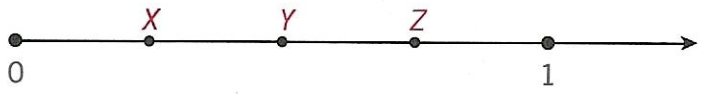

For each part of a set or region,
we can form a set of equivalent fractions.
$\{\frac{2}{3}, \frac{4}{6}, \ldots\}$
For each set of equivalent fractions, we think of exactly one **rational number.**
The rational number two thirds
With each rational number, we associate exactly one point on the number line.
0
1

The diagram shows the relationship of physical sets to sets of equivalent fractions, rational numbers, and points on the number line. Any fraction in a set of equivalent fractions can be used as a name for the rational number associated with that set. Usually, we use the lowest-terms fraction as the name for a rational number.

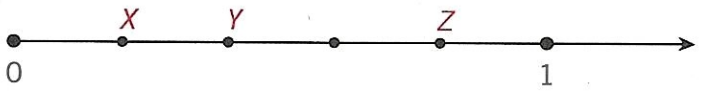
Exercises

1. Each set of equivalent fractions indicates one rational number. Give the correct point, *X*, *Y*, or *Z*, for the rational number.

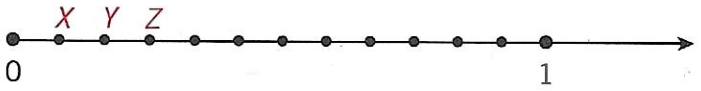
[a] $\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}$



[b] $\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \ldots\}$

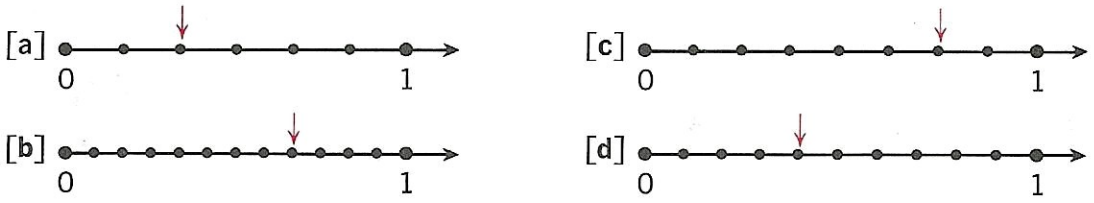


[c] $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots\}$

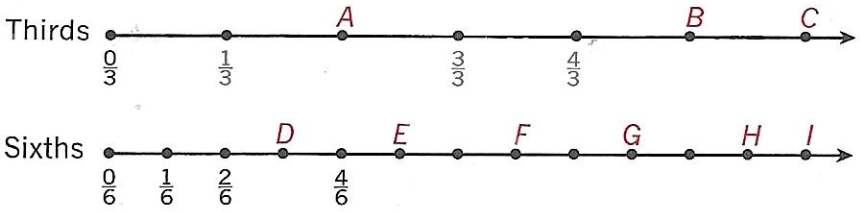


[d] $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$

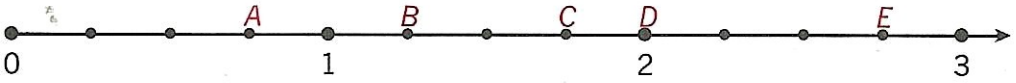
2. Give three fractions that name the rational number for the point indicated by the arrow on each number line.



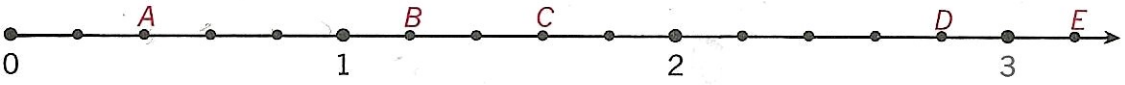
3. Use the denominator suggested at the left of each number line to give the fraction for the point indicated by each letter.



4. The number line is **scaled in fourths** because each segment between two whole-number points is divided into **four** congruent segments. Give the rational number for each lettered point.



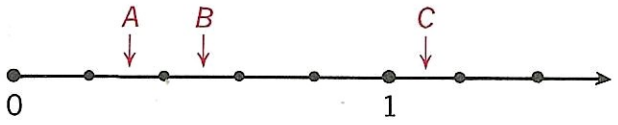
5. Give the rational number for each lettered point.



6. Draw a number line and scale it in thirds. Mark the points for the numbers $\frac{2}{3}$, $\frac{4}{3}$, $\frac{6}{3}$, $\frac{7}{3}$, and $\frac{8}{3}$.

7. Draw a number line and scale it in halves. Now mark additional points so that each of the small segments is divided into three equal parts. How is the number line now scaled?

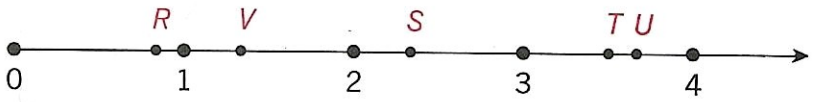
8. The arrows designate points halfway between points shown on the number line. What number goes with each lettered point?



9. Draw a number line and scale it in fourths. Mark the points $\frac{3}{4}$, $\frac{7}{4}$, $\frac{9}{4}$, and $\frac{12}{4}$ as A, B, C, and D respectively.

10. Scale the number line you used in Exercise 9 in eighths and give the fraction names for the points A, B, C, and D in eighths.

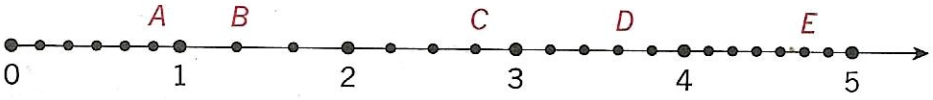
11. One of the five numbers $\frac{7}{3}$, $\frac{4}{5}$, $\frac{9}{7}$, $\frac{7}{2}$, $\frac{11}{3}$, is matched with each point on the number line. Match the numbers with the lettered points.



12. Draw a number line and mark the points for 0, 1, 2, 3, 4, and 5. Then mark the point for each number below.

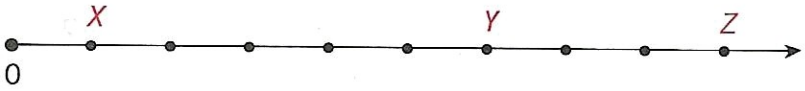
[a] $\frac{3}{2}$ [b] $\frac{12}{5}$ [c] $\frac{10}{3}$ [d] $\frac{25}{6}$ [e] $\frac{9}{2}$ [f] $\frac{1}{10}$ [g] $\frac{11}{10}$ [h] $\frac{21}{10}$

13. Give the number that goes with the points A, B, C, D, and E.



14. Give the number for each of the points X, Y, and Z if the number line is scaled in

[a] fourths [b] fifths [c] tenths [d] thirds [e] twelfths



Think

A set of three consecutive even numbers ⟶ {24, 26, 28}

A set of four consecutive odd numbers ⟶ {7, 9, 11, 13}

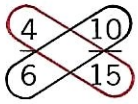
1. Find a set of three consecutive **even** numbers whose sum is 216.

2. Find a set of four consecutive **odd** numbers whose sum is 216.

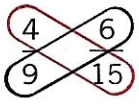
3. Find a set of three consecutive **whole** numbers whose sum is 216.

7-5 EQUALITY OF RATIONAL NUMBERS

Pete said that he knew an easy way to tell whether or not two fractions were equivalent. He used his method to show the class that $\frac{4}{6}$ was equivalent to $\frac{10}{15}$ and that $\frac{4}{9}$ was not equivalent to $\frac{6}{15}$. Here is what he wrote.



$\rightarrow$ $4 \cdot 15 = 6 \cdot 10$, so $\frac{4}{6} = \frac{10}{15}$

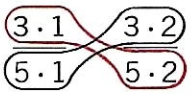


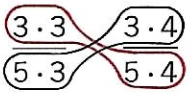
$\rightarrow$ $4 \cdot 15 \neq 9 \cdot 6$, so $\frac{4}{9} \neq \frac{6}{15}$

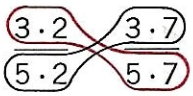
The class wondered whether Pete's method would always work. The teacher suggested that they look at some sets of equivalent fractions. She listed the set of fractions equivalent to $\frac{3}{5}$.

$$\left\{\frac{\mathbf{3}}{\mathbf{5}}, \frac{3 \cdot 1}{5 \cdot 1}, \frac{3 \cdot 2}{5 \cdot 2}, \frac{3 \cdot 3}{5 \cdot 3}, \frac{3 \cdot 4}{5 \cdot 4}, \frac{3 \cdot 5}{5 \cdot 5}, \frac{3 \cdot 6}{5 \cdot 6}, \frac{3 \cdot 7}{5 \cdot 7}, \ldots\right\}$$

Then Pete circled the factors as shown on the right and said, "The factors circled in color are the same as the factors circled in black. **This will be true no matter which pair of fractions I use.**"







Pete's check for the equivalence of fractions is important because

if two fractions are equivalent, then they name the same rational number.	if two fractions are not equivalent, they do not name the same rational number.

Exercises

1. Copy each exercise and insert the correct sign ($=$ or $\neq$) for each ◍.

[a] $\frac{6}{9}$ ◍ $\frac{4}{6}$ [b] $\frac{4}{10}$ ◍ $\frac{10}{25}$ [c] $\frac{20}{24}$ ◍ $\frac{5}{6}$

[d] $\frac{8}{9}$ ◍ $\frac{7}{8}$ [e] $\frac{10}{12}$ ◍ $\frac{12}{14}$ [f] $\frac{8}{21}$ ◍ $\frac{4}{10}$

[g] $\frac{7}{6}$ ◍ $\frac{21}{18}$ [h] $\frac{5}{1}$ ◍ $\frac{40}{8}$ [i] $\frac{3}{3}$ ◍ $\frac{11}{11}$

[j] $\frac{90}{100}$ ◍ $\frac{9}{10}$ [k] $\frac{11}{15}$ ◍ $\frac{99}{135}$ [l] $\frac{23}{31}$ ◍ $\frac{713}{961}$

2. Solve the equations.

[a] $\frac{2}{3} = \frac{n}{15}$ [c] $\frac{3}{n} = \frac{15}{40}$ [e] $\frac{6}{5} = \frac{24}{n}$ [g] $\frac{10}{12} = \frac{n}{18}$ [i] $\frac{n}{10} = \frac{40}{100}$

[b] $\frac{n}{4} = \frac{7}{2}$ [d] $\frac{16}{40} = \frac{n}{5}$ [f] $\frac{8}{12} = \frac{6}{n}$ [h] $\frac{5}{8} = \frac{n}{8}$ ★[j] $\frac{4}{n} = \frac{n}{9}$

3. Give the number for ***a***. Then give the number for ***b***.

[a] $2 \cdot 6 = 4 \cdot a \longrightarrow \frac{2}{4} = \frac{b}{6}$ [e] $3 \cdot 7 = a \cdot 21 \longrightarrow \frac{3}{b} = \frac{21}{7}$

[b] $3 \cdot a = 2 \cdot 12 \longrightarrow \frac{3}{2} = \frac{12}{b}$ [f] $3 \cdot 21 = a \cdot 7 \longrightarrow \frac{3}{b} = \frac{7}{21}$

[c] $1 \cdot 9 = a \cdot 3 \longrightarrow \frac{1}{b} = \frac{3}{9}$ [g] $20 \cdot 5 = 25 \cdot a \longrightarrow \frac{20}{25} = \frac{b}{5}$

[d] $12 \cdot 8 = 16 \cdot a \longrightarrow \frac{12}{16} = \frac{b}{8}$ [h] $18 \cdot a = 8 \cdot 9 \longrightarrow \frac{b}{9} = \frac{8}{18}$

4. For each equation, answer A, B, or C.

(A) There is exactly one whole number for ***n***.

(B) There is no whole number for ***n***.

(C) There are many whole numbers for ***n***.

[a] $\frac{0}{6} = \frac{n}{7}$ [c] $\frac{0}{6} = \frac{5}{n}$ [e] $\frac{n}{5} = \frac{5}{n}$ [g] $\frac{6}{n} = \frac{6}{7}$ [i] $\frac{2}{3} = \frac{2 \cdot n}{3 \cdot n}$

[b] $\frac{0}{n} = \frac{0}{8}$ [d] $\frac{n}{7} = \frac{7}{7}$ [f] $\frac{1}{1} = \frac{11}{n}$ [h] $\frac{3}{8} = \frac{n}{1}$

5. Match the fractions with the sets.

[a] $\frac{10}{70}$ [g] $\frac{18}{24}$

[b] $\frac{10}{15}$ [h] $\frac{26}{52}$

[c] $\frac{45}{72}$ [i] $\frac{42}{70}$

[d] $\frac{27}{36}$ [j] $\frac{8}{56}$

[e] $\frac{21}{35}$ [k] $\frac{54}{81}$

[f] $\frac{45}{90}$ [l] $\frac{60}{96}$

(A) $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \cdots\right\}$

(B) $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \cdots\right\}$

(C) $\left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \cdots\right\}$

(D) $\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \cdots\right\}$

(E) $\left\{\frac{1}{7}, \frac{2}{14}, \frac{3}{21}, \cdots\right\}$

(F) $\left\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \cdots\right\}$

6. Answer true (**T**) or false (**F**).

[a] $\frac{1}{3} = \frac{3}{9}$ [d] $\frac{7}{8} = \frac{56}{64}$ [g] $\frac{5}{9} = \frac{26}{45}$ [j] $\frac{63}{18} = \frac{49}{15}$

[b] $\frac{2}{5} = \frac{5}{2}$ [e] $\frac{2}{15} = \frac{12}{90}$ [h] $\frac{11}{12} = \frac{55}{60}$ ★[k] $\frac{7}{13} = \frac{91}{169}$

[c] $\frac{3}{7} = \frac{7}{13}$ [f] $\frac{5}{6} = \frac{45}{48}$ [i] $\frac{14}{16} = \frac{35}{80}$ ★[l] $\frac{11}{17} = \frac{187}{289}$

★**7.** Choose any four whole numbers. Form two fractions, using these four numbers in the following way: Select two of your numbers. Their product is the numerator of the first fraction. The product of the remaining pair of numbers is the denominator of the second fraction.

Now make another selection of two of your original numbers. Their product is the denominator of the first fraction. The product of the remaining pair is the numerator of the second fraction.

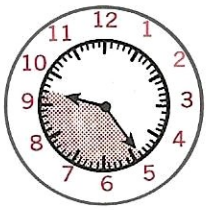
[a] Are the two fractions equivalent?

[b] Can you choose the numbers so that the two fractions will not be equivalent?

Think

What part of the clock face is shaded?

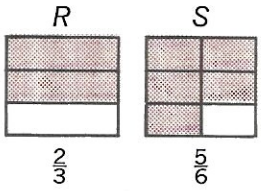
Time: 9:24



7-6 INEQUALITIES

We can compare two rational numbers in different ways.

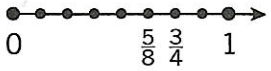
A We can compare parts of regions.



A smaller part of region R is shaded.

We write, $\frac{2}{3} < \frac{5}{6}$.

B We can use points on the number line.



The point for $\frac{3}{4}$ lies to the right of the point for $\frac{5}{8}$.

We write, $\frac{3}{4} > \frac{5}{8}$.

C We can choose fractions with the same denominator and then compare their numerators.

$$\frac{7}{8} = \frac{7 \cdot 6}{8 \cdot 6} = \frac{42}{48}$$

$$\frac{5}{6} = \frac{5 \cdot 8}{6 \cdot 8} = \frac{40}{48}$$

$42 > 40$

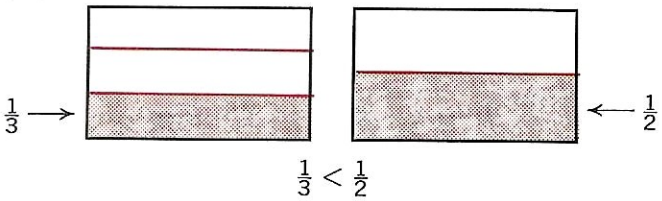
We write, $\frac{7}{8} > \frac{5}{6}$.

Method C is usually easier to use than either A or B. Notice that it is not necessary to choose the least common denominator for the fractions. Any common denominator will do.

Exercises

1. Compare the rational numbers by looking at parts of regions.

Example: $\frac{1}{3}, \frac{1}{2}$



$\frac{1}{3} < \frac{1}{2}$

[a] $\frac{1}{4}, \frac{1}{3}$ [b] $\frac{3}{5}, \frac{3}{8}$ [c] $\frac{3}{4}, \frac{4}{5}$ [d] $\frac{1}{2}, \frac{3}{5}$ ★[e] $\frac{7}{6}, \frac{6}{5}$

2. Use the method of points on a number line to compare the following rational numbers.

Example: $\frac{2}{5}, \frac{1}{4}$. First mark off the segment from 0 to 1 in fifths. Then mark it off, in color, in fourths.

$\frac{2}{5} > \frac{1}{4}$

[a] $\frac{1}{3}, \frac{1}{2}$ [b] $\frac{3}{4}, \frac{5}{8}$ [c] $\frac{2}{3}, \frac{3}{5}$ [d] $\frac{3}{4}, \frac{7}{8}$ ★[e] $\frac{8}{7}, \frac{7}{6}$

3. Give the numbers for ***a*** and ***b***. Then give the correct sign (> or <) for each ◍.

[a] $\frac{1}{4} = \frac{a}{12}$, $\frac{1}{3} = \frac{b}{12}$ → $\frac{1}{4}$ ◍ $\frac{1}{3}$

[b] $\frac{5}{3} = \frac{a}{6}$, $\frac{5}{2} = \frac{b}{6}$ → $\frac{5}{3}$ ◍ $\frac{5}{2}$

[c] $\frac{3}{4} = \frac{a}{36}$, $\frac{7}{9} = \frac{b}{36}$ → $\frac{3}{4}$ ◍ $\frac{7}{9}$

[d] $\frac{2}{5} = \frac{a}{35}$, $\frac{3}{7} = \frac{b}{35}$ → $\frac{2}{5}$ ◍ $\frac{3}{7}$

[e] $\frac{5}{8} = \frac{a}{48}$, $\frac{7}{12} = \frac{b}{48}$ → $\frac{5}{8}$ ◍ $\frac{7}{12}$

[f] $\frac{7}{6} = \frac{a}{30}$, $\frac{6}{5} = \frac{b}{30}$ → $\frac{7}{6}$ ◍ $\frac{6}{5}$

[g] $\frac{3}{10} = \frac{a}{100}$, $\frac{33}{100} = \frac{b}{100}$ → $\frac{3}{10}$ ◍ $\frac{33}{100}$

[h] $\frac{1}{3} = \frac{a}{90}$, $\frac{7}{30} = \frac{b}{90}$ → $\frac{1}{3}$ ◍ $\frac{7}{30}$

4. Copy each problem and use any method you wish to find the correct sign (< or >) for each ◍.

[a] $\frac{1}{3}$ ◍ $\frac{1}{2}$ [b] $\frac{1}{4}$ ◍ $\frac{3}{16}$ [c] $\frac{1}{4}$ ◍ $\frac{5}{16}$ [d] $\frac{3}{8}$ ◍ $\frac{2}{5}$ [e] $\frac{1}{9}$ ◍ $\frac{1}{10}$ [f] $\frac{2}{3}$ ◍ $\frac{5}{7}$ [g] $\frac{7}{4}$ ◍ $\frac{8}{5}$ [h] $\frac{3}{7}$ ◍ $\frac{4}{9}$ [i] $\frac{9}{10}$ ◍ $\frac{10}{11}$ [j] $\frac{2}{5}$ ◍ $\frac{3}{8}$ [k] $\frac{5}{12}$ ◍ $\frac{3}{8}$ [l] $\frac{5}{16}$ ◍ $\frac{1}{3}$ [m] $\frac{7}{8}$ ◍ $\frac{11}{12}$ [n] $\frac{19}{25}$ ◍ $\frac{77}{100}$ ★[o] $\frac{9}{11}$ ◍ $\frac{35}{43}$

5. Joe used the method of equal denominators to compare $\frac{5}{6}$ and $\frac{7}{8}$. Patty said that she could see a shortcut to Joe's work. Study Joe's work and Patty's work.

Joe's work

$\frac{5}{6} = \frac{5 \cdot 8}{6 \cdot 8}$

$40 < 42$

$\frac{7}{8} = \frac{7 \cdot 6}{8 \cdot 6}$

So $\frac{5}{6} < \frac{7}{8}$

Patty's work

$40 < 42$

So $\frac{5}{6} < \frac{7}{8}$

Use Patty's shortcut to decide which rational number in each pair is the larger.

[a] $\frac{4}{5}, \frac{7}{8}$ [b] $\frac{7}{10}, \frac{4}{5}$ [c] $\frac{5}{9}, \frac{7}{12}$ [d] $\frac{7}{6}, \frac{12}{10}$ [e] $\frac{12}{5}, \frac{5}{2}$ [f] $\frac{9}{10}, \frac{10}{9}$ [g] $\frac{1}{6}, \frac{1}{4}$ [h] $\frac{11}{15}, \frac{5}{7}$

Think

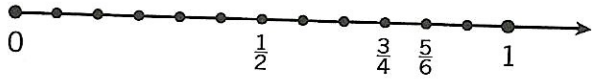
Try to arrange the nine letters in the nine squares so that each letter is moved one square to the right or left or one square above or below its original position.

If this can be done, show an arrangement of the letters. If you feel it cannot be done, tell why it is not possible.

A	B	C
D	E	F
G	H	I

7-7 ORDER OF RATIONAL NUMBERS

From the number line, we can see that $\frac{1}{2} < \frac{3}{4}$ and $\frac{3}{4} < \frac{5}{6}$.



We can shorten this statement and write: $\frac{1}{2} < \frac{3}{4} < \frac{5}{6}$.

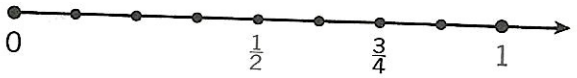
$\frac{1}{2} < \frac{3}{4} < \frac{5}{6}$

$\downarrow \quad \downarrow \quad \downarrow$

$\frac{6}{12} < \frac{9}{12} < \frac{10}{12}$

Discussion Exercises

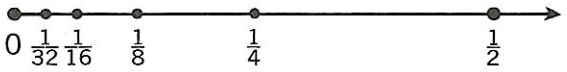
1. What is a name for the rational number for the point halfway between the points for $\frac{1}{2}$ and $\frac{3}{4}$?



2. Give a fraction for the ▒: $\frac{1}{2} < ▒ < \frac{3}{4}$.

3. [a] What is the smallest number in the set $\{\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$?

[b] Give the names for two rational numbers which are less than $\frac{1}{32}$.



Exercises

1. Write a short statement which has the same meaning as the given statement.

[a] $\frac{1}{2} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{8}$

[b] $\frac{1}{5} < \frac{3}{10}$ and $\frac{3}{10} < \frac{3}{8}$

[c] $\frac{5}{4} < \frac{5}{3}$ and $\frac{5}{3} < \frac{5}{2}$

[d] $\frac{1}{7} < \frac{1}{6}$ and $\frac{1}{6} < \frac{1}{5}$

[e] $\frac{1}{3} > \frac{1}{6}$ and $\frac{1}{6} > \frac{1}{12}$

[f] $\frac{2}{5} < \frac{3}{4}$ and $\frac{7}{8} > \frac{3}{4}$

2. Copy and complete each statement. Part a is completed for you.

[a] If $\frac{2}{3} < \frac{5}{6}$ and $\frac{5}{6} < \frac{7}{8}$, then $\frac{2}{3} < \frac{7}{8}$.

[b] If $\frac{1}{3} < \frac{1}{2}$ and $\frac{1}{2} < \frac{5}{8}$, then _?_.

[c] If $\frac{1}{1000} < \frac{1}{100}$ and $\frac{1}{100} < \frac{1}{10}$, then _?_.

[d] If $\frac{7}{3} < \frac{5}{2}$ and $\frac{5}{2} < \frac{11}{4}$, then _?_.

[e] If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then _?_.

3. If the number line is scaled in thirds and then the distance between each pair of points is divided into four equal parts, how is the number line now scaled?

4. Draw a number line and scale it in fifths. Divide the segments between each pair of points into three congruent segments. How is the number line now scaled?

5. Give the names for two rational numbers both of which are larger than $\frac{3}{5}$ but less than $\frac{4}{5}$.

6. Give the missing numerator for each $\square$.

[a] $\frac{3}{8} < \frac{\square}{16} < \frac{4}{8}$

[b] $\frac{1}{2} < \frac{\square}{8} < \frac{3}{4}$

[c] $\frac{6}{10} < \frac{\square}{20} < \frac{7}{10}$

[d] $\frac{5}{7} < \frac{\square}{14} < \frac{6}{7}$

[e] $\frac{4}{9} < \frac{\square}{18} < \frac{5}{9}$

[f] $\frac{2}{3} < \frac{\square}{24} < \frac{3}{4}$

[g] $\frac{0}{4} < \frac{\square}{8} < \frac{1}{4}$

[h] $\frac{7}{8} < \frac{\square}{48} < \frac{11}{12}$

7. What is the smallest rational number named by the fractions?

$$\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}$$

8. What is the largest rational number named by the fractions?

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$$

9. What is the smallest number named by the fractions?

$$\frac{2}{11}, \frac{2}{7}, \frac{2}{9}, \frac{2}{17}, \frac{2}{15}$$

10. Give the smallest and largest numbers named by the fractions.

$$\frac{3}{4}, \frac{3}{5}, \frac{5}{6}, \frac{2}{3}$$

★ **11.** What rational number is just as much larger than $\frac{2}{5}$ as it is smaller than $\frac{3}{5}$?

★ **12.** [a] Find the rational number for the point on the number line that is just halfway between the points for $\frac{1}{3}$ and $\frac{1}{5}$.

[b] Is $\frac{1}{4}$ larger or smaller than this number?

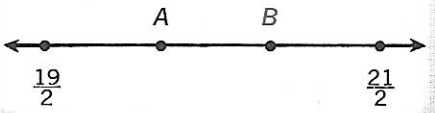
[c] Is the point for $\frac{1}{4}$ closer to the point for $\frac{1}{3}$ or the point for $\frac{1}{5}$?

★ **13.** If the numbers $\frac{3}{2}, \frac{3}{4}, \frac{7}{8}, \frac{7}{6}, \frac{15}{16}, \frac{5}{8}, \frac{2}{3}, \frac{19}{16}$ were listed in order of their closeness to 1, what would the order be?

★ **14.** Is there a smallest rational number that is still larger than zero?

Think

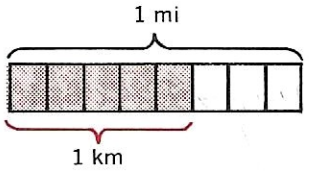
Give the lowest-terms fraction for each point A and B on the number line.



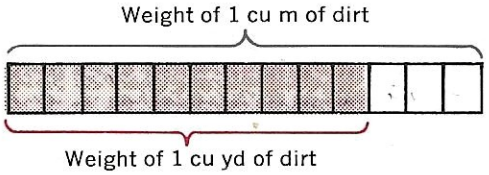
7-8 PROBLEMS—RATIONAL NUMBERS

Exercises

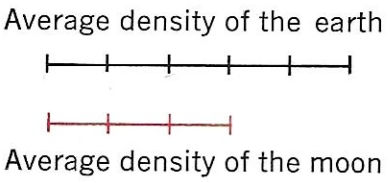
1. The figure compares a distance of 1 mi with a distance of 1 kilometer (km). Approximately what part of a mile is a kilometer?



2. The shaded region shows what fractional part the weight of 1 cu yd of dirt is of the weight of 1 cubic meter (cu m) of dirt. What is this fraction?



3. The average weight of the earth in pounds per cubic foot is greater than that of the moon. We say that the density of the moon is less than the density of the earth. The graph compares these densities. Express the average density of the moon as a fraction of the average density of the earth.



4. A cubic centimeter of gold is heavier than a cubic centimeter of iron. The graph compares the densities of these metals. Express the density of iron as a fraction of the density of gold.

5. In Mr. Wilson's seventh-grade mathematics class, there are 24 students. Of the students, 14 are girls. Give the lowest-terms fractions for the part of the class that is girls and the part of the class that is boys.

6. The length of each class period in school is 55 min. What part of an hour is this?

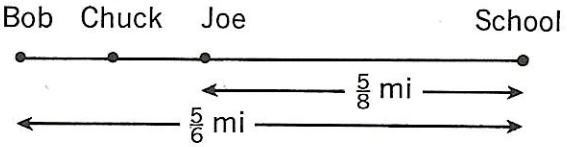
7. What part of one dollar is

[a] one cent? [c] one dime? [e] one half-dollar?
[b] one nickel? [d] one quarter? [f] three quarters

8. Dave has $\frac{3}{4}$ of a dollar. Don has $\frac{7}{10}$ of a dollar. Who has more money?

9. Mary lives 5 blocks from school. Joan lives 7 blocks from school. Write a fraction that compares the distance Mary lives from school with the distance Joan lives from school.

10. Bob lives $\frac{5}{6}$ mi west of his school. Joe lives on the same street $\frac{5}{8}$ mi west of the school. Chuck lives halfway between them. How far does Chuck live from school?

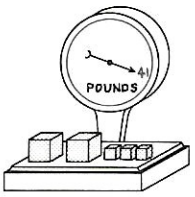


★**11.** In a certain orchard, there is 1 cherry tree for every 2 plum trees. For every plum tree there are 3 peach trees. Write a fraction which will compare the number of

[a] cherry trees to plum trees [c] cherry trees to peach trees.
[b] plum trees to peach trees

★**12.** Mr. Foss handed out rulers to the students in his mathematics class. He first handed out 2 rulers for every 5 students. Then he handed out 1 eraser for every 3 students. What fraction compares the number of articles (rulers and erasers) to the number of students?

Think



The small blocks weigh the same and the large blocks weigh the same. Each large block weighs 3 pounds more than two small blocks. How much does a large block weigh?

CHAPTER REVIEW

Key Concepts

1. **Fractions** are symbols used to compare sets.
2. Many different fractions may be suggested by one pair of sets or by the same region. Such fractions are called **equivalent fractions.**
3. A fraction is in lowest terms if the GCF of the numerator and denominator is 1.
4. The process of selecting the lowest-terms fraction for a given fraction is called **reducing the fraction to lowest terms.**
5. With each set of equivalent fractions, we associate exactly one **rational number.** Each rational number is paired with exactly one point on the number line.
6. If two fractions are equivalent, they name the same rational number. If two fractions are not equivalent, then they do not name the same rational number.
7. Different rational numbers correspond to different points on the number line.

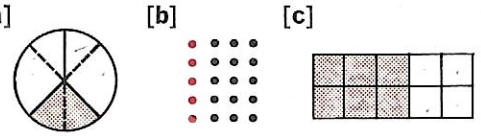
Vocabulary List

denominator	greatest common factor (GCF)	number line	region
equivalent fraction	inequality	numerator	scale
fraction	lowest-terms fraction	rational number	set

Review Exercises

1. Each figure suggests a pair of equivalent fractions. Give these fractions.

[a] [b] [c]



2. Give three fractions, each of which is equivalent to the fraction given.

[a] $\frac{4}{5}$ [b] $\frac{1}{6}$ [c] $\frac{5}{8}$ [d] $\frac{3}{10}$ [e] $\frac{5}{16}$

3. Give the GCF of the numerator and denominator of each fraction.

[a] $\frac{12}{15}$ [c] $\frac{9}{27}$ [e] $\frac{26}{39}$ [g] $\frac{25}{30}$

[b] $\frac{8}{12}$ [d] $\frac{8}{9}$ [f] $\frac{14}{21}$ [h] $\frac{80}{90}$

4. Reduce each fraction to lowest terms.

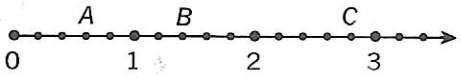
[a] $\frac{16}{20}$ [e] $\frac{20}{24}$ [i] $\frac{12}{9}$

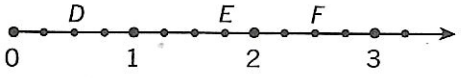
[b] $\frac{10}{12}$ [f] $\frac{16}{28}$ [j] $\frac{48}{100}$

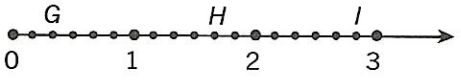
[c] $\frac{24}{36}$ [g] $\frac{2 \cdot 5}{7 \cdot 5}$ [k] $\frac{9}{30}$

[d] $\frac{16}{40}$ [h] $\frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 5}$ [l] $\frac{60}{72}$

5. Give the lowest-terms fraction for each letter.

[a]


[b]


[c]


6. Solve the equations.

[a] $\frac{4}{5} = \frac{n}{20}$ [e] $\frac{9}{10} = \frac{n}{100}$

[b] $\frac{3}{4} = \frac{n}{16}$ [f] $\frac{2}{3} = \frac{n}{27}$

[c] $\frac{1}{2} = \frac{n}{30}$ [g] $\frac{5}{8} = \frac{n}{40}$

[d] $\frac{3}{16} = \frac{n}{64}$ [h] $\frac{5}{12} = \frac{n}{60}$

7. Copy and give the correct sign ($<$, $=$, or $>$) for each ▒.

[a] $\frac{5}{8}$ ▒ $\frac{10}{12}$ [d] $\frac{3}{7}$ ▒ $\frac{4}{9}$

[b] $\frac{4}{3}$ ▒ $\frac{9}{6}$ [e] $\frac{5}{6}$ ▒ $\frac{4}{5}$

[c] $\frac{3}{2}$ ▒ $\frac{9}{6}$ [f] $\frac{5}{9}$ ▒ $\frac{4}{7}$

8. Copy and correctly fill in each screen.

[a] If $\frac{3}{4} < \frac{5}{6}$ and $\frac{5}{6} < \frac{7}{8}$, then $\frac{3}{4} <$ ▒.

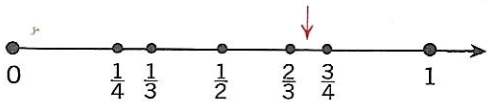
[b] If $\frac{1}{2} > \frac{1}{4}$ and $\frac{1}{4} >$ ▒, then $\frac{1}{2} > \frac{1}{8}$.

[c] If ▒ $> \frac{2}{4}$ and $\frac{2}{4} > \frac{2}{5}$, then $\frac{2}{3} > \frac{2}{5}$.

[d] The statement, $\frac{5}{6} > \frac{2}{3} > \frac{5}{8}$, means $\frac{5}{6} > \frac{2}{3}$ and ▒ $>$ ▒.

[e] $\frac{3}{4} <$ ▒ $< \frac{5}{6}$

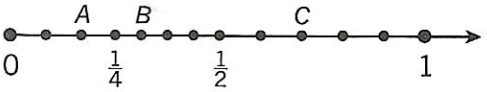
9. Give the rational number for the point that is halfway between the points for $\frac{2}{3}$ and $\frac{3}{4}$.



10. Give a whole number for n: $\frac{1}{5} > \frac{n}{60} > \frac{1}{6}$.

11. Mr. Knowlton advertises cider at 48 cents per half gallon. Another grocer, Mr. Shoup, shows a sale price of five pints for 55 cents. What is the fraction that compares Mr. Knowlton's price per quart of cider to Mr. Shoup's price per quart? Which grocer is offering the better sale?

★**12.** Study how the number line is scaled and then give the rational numbers for *A*, *B*, and *C*.



CHAPTER SUPPLEMENT

Reading List

ADLER, IRVING. *The Giant Golden Book of Mathematics.* New York, Golden Press, 1960; pp. 42–43.

———. *The New Mathematics.* New York, The New American Library of World Literature, 1959; pp. 73–81.

KASNER, EDWARD, and NEWMAN, JAMES. *Mathematics and the Imagination.* New York, Simon and Schuster, 1940; pp. 47–53.

MIDONICK, HENRIETTA O. (ed.). *The Treasury of Mathematics.* New York, Philosophical Library, 1965; pp. 46–55.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS. *Twenty-seventh Yearbook*. Washington, D.C., 1963; pp. 221–226.

VAN DER WAERDEN, B. L. *Science Awakening*. New York, John Wiley and Sons, 1963; pp. 19–23, 48–51.

Activities

A. Read about how the Egyptians used "unit fractions." Then explain their method to your class by giving examples of your own. (See *Science Awakening* and *Twenty-seventh Yearbook*.)

B. Estimate the dimensions of some objects in your classroom (such as the chalkboards, doors, windows, or the room itself). Then measure these objects to the nearest $\frac{1}{4}$ inch in order to determine how accurate your estimates were.

C. Find a recipe for soup that will serve eight people. Then determine how much of each ingredient you will need if you make the soup for four people and then for twelve people.

D. Read about the Babylonian notation for fractions and then explain this notation to your class. (See *The Treasury of Mathematics*, pp. 51–55.)

TO BE

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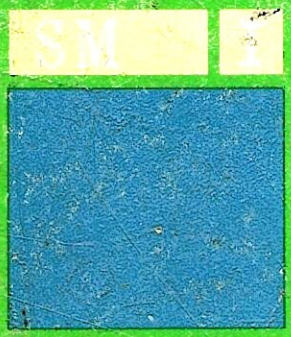
EICHOLZ • O'DAFFER • BRUMFIEL • SHANKS • FLEENOR